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A. KOSTOVSKII

GEOMETRICAL CONSTRUCTIONS WITH COMPASSES ONLY

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Geometrical constructions with compasses only

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. Н. Костовский

ГЕОМЕТРИЧЕСКИЕ ПОСТРОЕНИЯ ОДНИМ ЦИРКУЛЕМ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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A. Kostovskii

GEOMETRICAL CONSTRUCTIONS WITH COMPASSES ONLY

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by
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PREFACE

The author of this book has often given lectures on the theory of geometrical constructions to participants in mathematical olympiads, which have been organized every year since 1947, for the pupils of secondary schools in the city of Lvov. The first chapter of this work is based on these lectures.

The second chapter describes the investigations made by the author in connection with geometrical constructions carried out by compasses alone with a bounded opening of its legs.

This book is designed for a wide circle of readers. It should help teachers and pupils of senior classes of secondary schools to acquaint themselves in greater detail with geometrical constructions carried out by compasses alone. This work can serve as a teaching aid in school mathematical clubs. It can also be used by students of physical and mathematical departments of universities and teachers' training colleges to deepen their knowledge of elementary mathematics.

The author would like to express his sincere gratitude to professor A. N. Kovan'ko, assistant professors V. F. Rogachenko and I. F. Teslenko, and to experienced teacher B. G. Orach for reading the manuscript and offering a great deal of valuable advice.

INTRODUCTION

Geometrical constructions form a substantial part of a mathematical education. They represent a powerful tool of geometrical investigations.

The tradition of limiting the tools of geometrical constructions to a ruler and compasses goes back to remote antiquity. The famous geometry of Euclid (3rd century B.C.) was based on geometrical constructions carried out using a pair of compasses and a ruler, the compasses and ruler being regarded as equivalent instruments; it did not matter whether the construction was carried out with a pair of compasses alone, or with a ruler alone, or with both a pair of compasses and a ruler.

It was noted a long time ago that compasses are a more precise tool than rulers. Certain constructions could be carried out by compasses, without using a ruler; for example, dividing a circumference into six equal parts, constructing a point symmetric to a given point with respect to a given straight line, and so on. Attention was drawn to the fact that in engraving thin metal plates, in marking out indexing dials of astronomical instruments only compasses are used. This, probably, was the stimulus for investigations into geometrical constructions that could be carried out by compasses alone.

In 1797 the Italian mathematician Lorenzo Mascheroni, a professor of the University of Pavia, published an extensive tract called *The Geometry of Compasses*, which was later translated into French and German. In the tract he proved the proposition that

All construction problems solvable by means of a pair of compasses and a ruler can also be solved exactly by a pair of compasses alone.

This statement was proved in 1890 by Adler in an original way, using inversion. He also proposed a general method of solving geometrical construction problems by means of compasses alone. In 1928 the Danish mathematician Hjelmslev discovered in a bookshop in Copenhagen a book by G. Mohr titled *The Danish Euclid* and published in 1672 in Amsterdam. In the first part of the book there was a complete solution of Mascheroni's problem. Thus, it had been shown

a long time before Mascheroni that all geometrical constructions capable of being carried out by a pair of compasses and a ruler can also be carried out by a pair of compasses alone.

The branch of geometry dealing with constructions which can be completed using a pair of compasses alone is called the *geometry of the compasses.*

In 1833 the Swiss geometer Jacob Steiner published a book called *Geometrical Constructions Using a Straight Line and a Fixed Circle*, in which he investigated constructions carried out by a ruler alone. His basic result can be formulated as follows:

Every construction problem solvable by compasses and a ruler can also be solved by using a ruler alone given a circle with fixed centre and radius in the plane of the drawing.

Thus, in order to make the ruler equivalent to the compasses, it is sufficient to use a pair of compasses once.

The Russian mathematician Lobachevskii introduced in the early 19th century a new geometry which later became known as non-Euclidean or Lobachevskian geometry. Recently, thanks to the efforts of many scholars, especially Soviet ones, the theory of geometrical constructions in Lobachevskian plane has been rapidly developed.

A. S. Smogorzhevskii, V. F. Rogachenko, K. K. Mokrishchev, among others, have investigated constructions in the Lobachevskian plane without a ruler and shown the possibility of executing constructions similar to those of Mascheroni in the Euclidean plane.

Soviet scientists have now completely and rigorously formulated a theory for the geometrical constructions in the Lobachevskian plane, a theory as complete as the theory of geometrical constructions in the Euclidean plane.

1. CONSTRUCTIONS BY COMPASSES ONLY

Sec. 1. On the Possibility of Solving Geometrical Construction Problems by Compasses Only. The Basic Theorem

In this section we give the proof of the basic theorem of the geometry of compasses for which purpose it is necessary to examine the solutions of problems on construction by compasses alone.

It is clear that we cannot draw a continuous straight line given by two points using only compasses, although as will be shown later we can construct one, two, and, generally, any number of points on a given straight line*. Thus, the Mohr-Mascheroni theory does not cover the entire construction of a straight line.

In the geometry of compasses, a straight line or a segment is defined by two points and is not considered a continuous straight line (drawn with a ruler). *The construction of a straight line is said to be completed if any two of its points are constructed.*

We introduce the notation.

(AB) is a straight line passing through points A and B,

$[AB]$ is a segment AB,

$|AB|$ is the distance between the points A and B,

(O, r) is a circumference (or a circle) with centre O and radius r,

$(A, |BC|)$ is a circumference (or a circle) with centre A and radius $r = |BC|$.

Let us agree to write the phrase "With point O as centre and radius r we describe a circle (or draw an arc)" in the short form: "We describe (or draw) the circle (O, r)", or sometimes still shorter: "We describe (O, r)". The phrase "The segment AB, where $|AB| = a$" we consider to be equivalent to "The segment a", and, accordingly, if $|CD| = n|AB|$ we may say that the segment CD is n times as great as the segment AB.

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* From the practical point of view, there is no ground to regard a straight line as constructed if some of its points are constructed.

Other symbols and notation used in this book are given in Appendix 1.

Problem 1. Construct a point symmetric to a given point C with respect to a given straight line AB.

Given (AB) and point C. Construct $C_1 = S_{(AB)}(C)$*.

Construction. We describe the circles $(A, |AC|)$ and $(B, |BC|)$ which intersect at point C_1 (Fig. 1). The point C_1 is the required point.

If the point C lies on the straight line AB, then it is symmetric to itself [i.e. $C = S_{(AB)}(C)$].

Note. To verify that three given points A, B, and X lie on the same straight line, it is necessary to construct any point

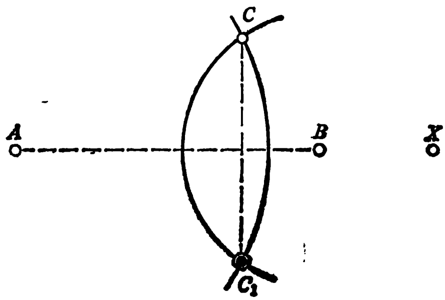

Fig. 1

C outside the straight line AB and then the point C_1 symmetric to C. Obviously, the point X lies on the straight line AB if and only if $|CX| = |C_1X|$.

Problem 2. Construct a segment 2, 3, 4, . . ., and in general n times as great as a given segment AA_1 (n is any natural number, $n \in \mathbf{N}$).

Given $|AA_1| = r$. Construct $[AA_n]$, $|AA_n| = n\,|AA_1|$, where $n \in \mathbf{N}$.

Construction (1st method). Keeping the opening of the compasses constant and equal to r, we describe the circle (A_1, r). Then we construct the point A_2 diametrically opposite to the point A, for which purpose we describe the circles (A, r), (B, r) and (C, r), at the intersection points of

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* See Appendix 1.

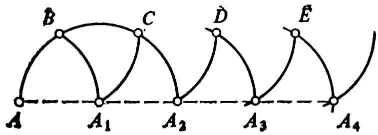


Fig. 2

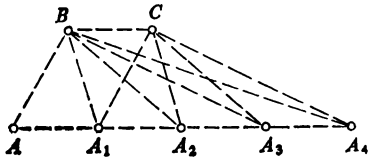


Fig. 3

these circles with the circle (A_1, r) we obtain the points B, C, and A_2. The segment $|AA_2| = 2r$ (Fig. 2).

Now we describe the circle (A_2, r) which intersects the circle (C, r) at the point D. At the intersection point of the circles (A_2, r) and (D, r) we obtain the point A_3. The segment $|AA_3| = 3r$, and so on.

Having carried out the above construction n times, we get the segment $|AA_n| = nr$.

The validity of the result follows from the fact that compasses with an opening equal to the radius of a circle divide its circumference into six equal parts.

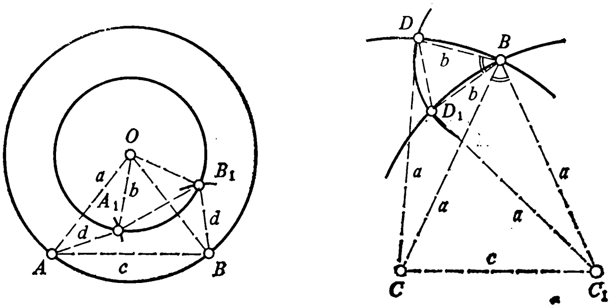
Construction (2nd method). We take an arbitrary point B outside the straight line AA_1 and draw the circles $(A_1, |AB|)$ and (B, r) which intersect at the point C (Fig. 3). If the circles (A_1, r) and $(C, |A_1B|)$ are drawn, they will intersect at the sought-for point A_2. The segment $|AA_2| = 2r$. We describe the circles (A_2, r) and $(C, |A_2B|)$ and denote their intersection point by A_3. Here $|AA_3| = 3r$, and so on.

The validity of the result follows immediately from the fact that the figures $ABCA_1$, A_1BCA_2, A_2BCA_3, ... are parallelograms.

Note. It is also easy to construct segments 2, 4, 8, 16, ..., 2^k times as great as the given segment AA_1. To this end we describe the circle (A_1, r) and find the point A_2 diametrically opposite to the point A ($|AA_2| = 2r$), describe the circle $(A_2, 2r)$ and find the point A_4 diametrically oppo-

site to the point A ($|AA_4| = 4r$). The point A_8, which is diametrically opposite to the point A on the circle $(A_4, 4r)$, defines $|AA_8| = 8r$, and so on. After k steps we obtain $|AA_{2^k}| = 2^k r$*.

Problem 3. Construct a segment whose value x is the extreme term of the proportion $a/b = c/x$, where a, b, and c are the values of the given segments.



a b

Fig. 4

Given the segments a, b, and c. Construct a segment x such that $a/b = c/x$.

Construction (1st method). We take an arbitrary point O and describe two circles (O, a) and (O, b). With an arbitrary point A on the circle (O, a) as the centre we describe (A, c) and denote the intersection point of these circles by B. Now we describe two circles (A, d) and (B, d) of arbitrary radius $d > |a - b|$, which intersect (O, b) at the points A_1 and B_1. The segment $x = |A_1B_1|$ is the required segment (Fig. 4, a).

Proof. $\triangle AOA_1 \cong \triangle BOB_1$ (three sides are equal), therefore $\widehat{AOA_1} = \widehat{BOB_1}$ and $\widehat{AOB} = \widehat{A_1OB_1}$. The isosceles triangles AOB and A_1OB_1 are similar, hence,

$$a/b = c/|A_1B_1|.$$

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* It is easy to show how to construct segments $m, m^2, m^3, \ldots, m^k$ times as great as the given segment, where $m = 3, 4, 5, \ldots$. For example, for $m = 5$ we construct the segment $|AA_5| = 5|AA_1|$ (Problem 2). Given the segment AA_5, we construct the segment $|AA_{25}| = 5|AA_5| = 5^2|AA_1|$ (Problem 2). Then we construct the segment $|AA_{125}| = 5|AA_{25}| = 5^3|AA_1|$, and so on.

The construction given above is possible for $c < 2a$. If $c \geqslant 2a$ and $b < 2a$, we construct a segment whose value is the extreme term of the proportion $a/c = b/x$. In the case of $c \geqslant 2a$ and $b \geqslant 2a$ we construct the segment na (Problem 2) taking n such that $c < 2na$* (or $b < 2na$). Then we construct a segment y whose value is the extreme term of the proportion $na/b = c/y$. If now we construct a segment $x = ny$ (Problem 2), then we obtain a segment which is the fourth proportional to the segments a, b, and c.

In fact

$$na/b = c/y \text{ or } a/b = c/ny.$$

Construction (2nd method). We construct the circles (C, a) and (C_1, a), where C and C_1 are the end points of the segment c. At the intersection we obtain the point B (Fig. 4,b). The circle (B, b) intersects the circles (C, a) and (C_1, a) at points D and D_1. The segment DD_1 is the required segment.

Proof. The isosceles triangles C_1BD_1 and BCD are congruent, therefore $\widehat{CBD} = \widehat{C_1BD_1}$. Hence, $\widehat{CBC_1} = \widehat{DBD_1}$. From the fact that the isosceles triangles CBC_1 and DBD_1 are similar it follows that

$$a/b = c/|DD_1|.$$

For $c \geqslant 2a$ and $b \geqslant 2a$, as well as in the first case, we find the segment na such that $2na > c$ and $2na > b$ and then construct a segment y that is the extreme term of the proportion $na/b = c/y$. The segment ny is the required segment.

Problem 4. Bisect the arc AB of the circle (O, r).

We can assume that the centre O of the circle is known; it will be shown below (Problem 13) how to construct the centre of a circle (or arc) using only compasses.

Construction. Putting $a = |AB|$, we describe circles (O, a), (A, r) and $(B\ r)$; at the intersection we obtain the points

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* We find a segment $2na > c$ in the following way. We construct the segment $a_1 = 2a$ (Problem 2). We describe a circle (O_1, c) with an arbitrary point O_1 as centre and lay off in an arbitrary direction segments $|O_1A_1| = a_1$, $|O_1A_2| = 2a_1$, $|O_1A_3| = 3a_1$, and so on (Problem 2). After a finite number of steps we arrive at the point A_n which lies outside (O_1, c). Obviously, the segment $|O_1A_n| = na_1 = 2na > c$.

C and D (Fig. 5). At the intersection of the circles $(C, |CB|)$ and $(D, |AD|)$ we obtain the point E. If now we draw the circles $(C, |OE|)$ and $(D, |OE|)$, then, at their intersection, we obtain the points X and X_1. The point X bisects the arc AB, while the point X_1 bisects the arc which forms, together with the first one, the full circle. (In the case

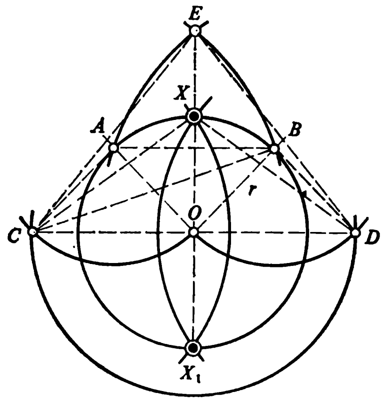

Fig. 5

the whole circle (O, r) is drawn, we can draw only one circle (either $(C, |OE|)$ or $(D, |OE|)$) which defines points X and X_1, when intersecting (O, r)).

Proof. The figures $ABOC$ and $ABDO$ are parallelograms, therefore the points C, O, and D lie on the same straight line ($[CO] \parallel [AB]$, $[OD] \parallel [AB]$). From the fact that the triangles CED and CXD are isosceles it follows that $\widehat{COE} = \widehat{COX} = 90°$. Thus the segment OX is perpendicular to the chord AB. Consequently, in order to prove that the point X bisects the arc AB, it is sufficient to show that $|OX| = r$.

Since $ABOC$ is a parallelogram, we have

$$|AO|^2 + |BC|^2 = 2|OB|^2 + 2|AB|^2$$

or

$$r^2 + |BC|^2 = 2r^2 + 2a^2,$$

so that

$$|BC|^2 = 2a^2 + r^2.$$

Since the triangle COE is right-angled, we have

$$|CE|^2 = |BC|^2 = |OC|^2 + |OE|^2,$$

whence

$$2a^2 + r^2 = a^2 + |OE|^2$$

and

$$|OE|^2 = a^2 + r^2.$$

Finally, using the right-angled triangle COX, we obtain

$$|OX| = \sqrt{|CX|^2 - |OC|^2} = \sqrt{|OE|^2 - |OC|^2}$$
$$= \sqrt{a^2 + r^2 - a^2} = r.$$

This construction is also valid when the given arc $\overset{\frown}{AB}$ is a semicircle ($\overset{\frown}{AB} = 180°$). Here the points A and B lie on the segment CD and the circles (A, r) and (B, r) touch the circle (O, a) at the points C and D, respectively. Because draftsman's instruments (compasses) are imperfect, it is difficult to indicate the position of the points C and D exactly. In this case ($\overset{\frown}{AB} = 180°$) it is necessary to bisect the arc $\overset{\frown}{A_1B_1}$ ($\overset{\frown}{A_1B_1} \neq 180°$) such that $\overset{\frown}{AA_1} = \overset{\frown}{BB_1} > 0$ and $\overset{\frown}{AA_1} + \overset{\frown}{A_1B_1} + \overset{\frown}{B_1B} = \overset{\frown}{AB}$. Obviously, the point bisecting the arc A_1B_1 will also bisect the arc AB.

As we have already pointed out, in the geometry of the compasses a straight line is regarded to be constructed as soon as any two of its points are defined. In the subsequent discussion (Problems 22, 23, 24, and others) we are going to construct one, two, and, in general, any number of points on the given straight line using compasses alone. This can be done in the following way.

Problem 5. Construct one or several points on a straight line, defined by two points A and B.

Given (AB). Construct $X \in (AB)$, $X_1 \in (AB)$, ...

Construction. We take an arbitrary point C outside the straight line AB (Fig. 6) and construct a point C_1 symmetric to C with respect to AB (Problem 1). We describe the circles (C, r) and (C_1, r) of arbitrary radius r. At their intersection we obtain the required points X and X_1, which lie on the straight line AB. Varying the radius r, it is possible to construct any number of points on the given straight line: X', X'_1, etc.

Proof. The point C_1 is symmetric to the point C, therefore the straight line AB passes through the midpoint of the segment CC_1 at right angles. Therefore the straight line AB

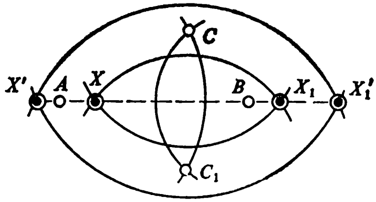

Fig. 6

is a set of points equidistant from the points C and C_1. By virtue of construction $|CX| = |C_1X| = r$ and $|CX_1| = |C_1X_1| = r$, hence, $X \in (AB)$ and $X_1 \in (AB)$.

Problem 6. Construct the intersection points of the circle (O, r) and the straight line given by two points A and B.

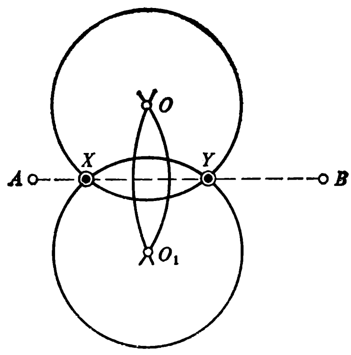

Fig. 7

Given (O, r) and (AB). Construct $\{X; Y\} = (O, r) \cap (AB)$.

Construction when the centre O does not lie on the straight line AB* (Fig. 7).

We construct the point O_1 symmetric to the centre O of the given circle with respect to the straight line AB (Prob-

* With the help of compasses alone it is easy to check whether three given points lie on one straight line or not (see note to Problem 1).

lem 1). We describe the circle (O_1, r) which intersects the given circle at the required points X and Y.

Proof. It was shown in the previous problem that the points X and Y lie on the straight line AB. These points also belong to the given circle (O, r), hence, $\{X; Y\} = (O, r) \cap \cap (AB)$.

Construction when the centre O of the given circle lies on the straight line AB (Fig. 8).

We describe the circle (A, d) of arbitrary radius d so that it intersects the given circle at two points C and D. We

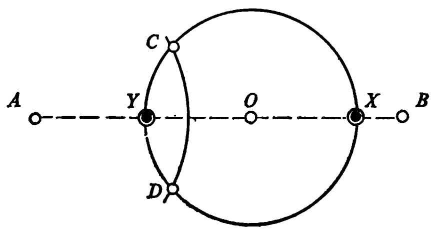

Fig. 8

halve the arcs CD of the given circle (O, r) (Problem 4). The points X and Y are the required ones.

Note. Given two segments a and r, where $a = |AO|$, we describe the circle (O, r) and do the above construction again. As a result we obtain the sum and difference of the lengths of the given segments:

$$|AX| = |AO| + |OX| = a + r \quad \text{and} \quad |AY| = |AO| - |OX| = a - r.$$

Problem 7. Construct the intersection point of two straight lines AB and CD, each of which is given by two points.

Given (AB) and (CD). Construct the point $X = (AB) \cap \cap (CD)$.

Construction. We construct points C_1 and D_1 symmetric to points C and D, respectively, about the given straight line AB (Fig. 9). We describe the circles $(D_1, |CC_1|)$ and $(C, |CD|)$ and denote the point of their intersection by E. We construct the segment x, which is the extreme term of the proportion $|DE|/|DD_1| = |CD|/x$ (Problem 3). At the intersection of the circles (D, x) and (D_1, x) we get the required point X.

Proof. Since the point C_1 is symmetric to the point C and the point D_1 is symmetric to the point D, we obviously find the intersection point of the given straight lines if we construct the point of intersection of the straight lines CD and C_1D_1.

The figure CC_1D_1E is a parallelogram, consequently, the points D, D_1 and E lie on the same straight line (because

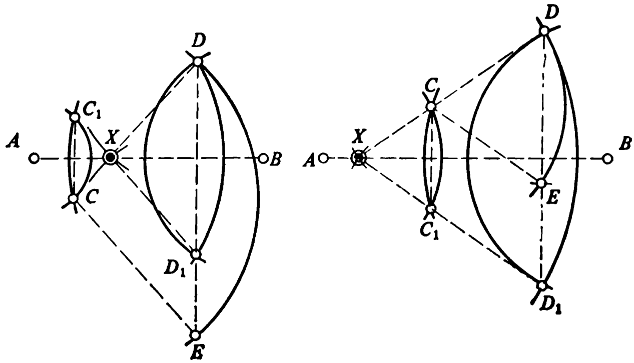

Fig. 9

$(DE) \parallel (CC_1)$ and $(DD_1) \parallel (CC_1)$). The triangles CDE and XDD_1 are similar, therefore

$$|DE|/|DD_1| = |CE|/|D_1X|,$$

but

$$|CE| = |CD| = |C_1D_1|.$$

The segment $x = |D_1X|$ is the extreme term of the proportion $|DE|/|DD_1| = |CD|/x$.

Note. The given straight lines AB and CD are parallel if and only if $|CC_1| = |DD_1|$, where the points C_1 and D_1 are symmetric to C and D, respectively, about the straight line AB.

* * *

Now we can show the validity of the basic theorem of the geometry of compasses (the Mohr-Mascheroni theorem).

Each problem on construction by compasses and a ruler in the Euclidean plane is always reducible to the solution of the following very simple basic problems arranged in the following order:

1. Draw a straight line through two given points.
2. Describe a circle of a given radius with a given point as centre.
3. Find the intersection points of two given circles.
4. Find the intersection points of a given circle with a straight line given by two points.
5. Find the intersection point of two straight lines, each of which is given by two points.

In order to prove that any construction problem which can be solved with a ruler and compasses can also be solved by compasses alone, it is sufficient to show that all these basic operations can be carried out by compasses alone.

The second and third operations can be done directly by compasses. The remaining basic operations were presented in Problems 5-7.

Suppose that a certain construction problem, solvable by compasses and a ruler, has to be solved by compasses alone. Let us imagine this problem solved by a ruler and compasses. As a result, the solution is reduced to a certain sequence of the five basic operations. Having carried out each of these operations by compasses alone (Problems 5-7), we arrive at the solution of the original problem.

The method of solving geometrical construction problems by compasses alone results, as a rule, in quite complicated and lengthy manipulations, being therefore inefficient. But from the theoretical point of view the method makes it possible to show the validity of the following basic theorem of the geometry of compasses.

The basic theorem. *All construction problems, solvable by compasses and a ruler, can be solved exactly by compasses alone.*

Sec. 2. Solution of Geometrical Construction Problems by Compasses Only

In this section we discuss the solution of interesting problems in the geometry of compasses, which were mainly worked up by Mohr, Mascheroni, and Adler. The solutions of some of these problems will be used in Chapter 2.

Problem 8. Draw a straight line perpendicular to a given segment AB and passing through one of its end points.

Given $[AB]$. Construct $(AE) \perp [AB]$.

Construction (1st method). Keeping the opening of the compasses constant and equal to an arbitrary value r, we draw the circles (A, r) and (B, r) until they meet at the point O. (We take any of two intersection points of these circles). We describe the circle (O, r) and construct the

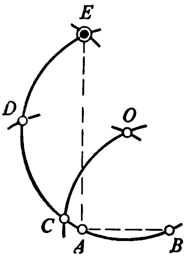

Fig. 10

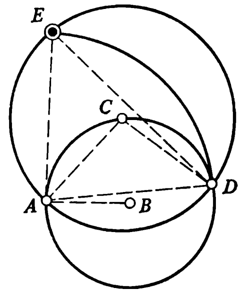

Fig. 11.

point E on it, which is diametrically opposite to the point B (see Problem 2). The straight line AE is the required one (Fig. 10), i.e. $(AE) \perp [AB]$.

The validity of the construction follows from the fact that the angle BAE is inscribed in the circle (O, r) and is subtended by its diameter.

Construction (2nd method). We describe the circle $(B, |AB|)$ (Fig. 11), take an arbitrary point C on it, and draw the circle $(C, |AC|)$. Let D be the intersection point of these circles. If now a third circle $(A, |AD|)$ is drawn which intersects the circle $(C, |AC|)$ at the point E, then (AE) is perpendicular to $[AB]$, i.e. (AE) is the required straight line.

Proof. The segment AC joins the centres of the circles $(A, |AD|)$ and $(C, |AC|)$ while DE is their common chord. This means that (AC) is perpendicular to $[DE]$ and $\widehat{CAD} = \widehat{CAE}$ (the triangle ADE is isosceles).

On the other hand, $\widehat{CAD} = \widehat{ADC} = 0.5\ \overset{\smile}{AC}$.

It follows from the last equalities that

$$\widehat{CAE} = \frac{1}{2}\ \overset{\smile}{AC}.$$

Therefore the straight line AE is the tangent to the circle $(B, |AB|)$ at the point A, so that (AE) is perpendicular to $[AB]$.

Problem 9. Construct a segment n times smaller than a given segment AB (in other words divide a segment AB into n equal parts, $n = 2, 3, \ldots$). $|AB| = a$.

Given $[AB]$ and $n \in \mathbf{N}$. Construct $[AX]$, $|AX| = \frac{1}{n}|AB|$.

Construction (1st method). We construct the segment $|AC| = n \cdot |AB|$ (Problem 2). We describe the circle

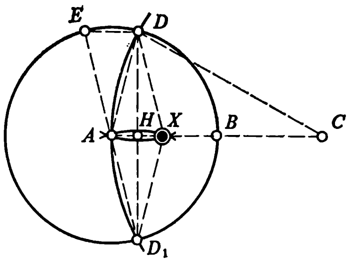

Fig. 12

$(C, |CA|)$ which intersects the circle (A, a) at the points D and D_1. If now we describe the circles (D, a) and (D_1, a) and denote the point of their intersection by X, then we obtain the required segment AX (Fig. 12).

The point X lies on the straight line AB. Increasing the segment AX 2, 3, and so on, $n - 1$ times (Problem 2), we construct the points which divide the segment AB into n equal parts.

Proof. From the similarity of the isosceles triangles ACD and ADX (the angle A is common) it follows that

$$|AC|/|AD| = |AD|/|AX|$$

or

$$|AD|^2 = a^2 = |AC| \cdot |AX| = na \cdot |AX|.$$

Hence

$$|AX| = \frac{1}{n}|AB| = \frac{1}{n}a.$$

Note. For large values of n the point X is poorly defined because the arcs of the circles (D, a) and (D_1, a) intersect at

the point X at a very small angle*. In this case, to find the point X accurately, instead of the circle (D_1, a) we can draw the circle $(A, |DE|)$, where E is the point diametrically opposite to the point D_1 of the circle (A, a).

Construction (2nd method). We construct the segment $|AC| = n \cdot |AB|$ (Problem 2). Then we describe the circles

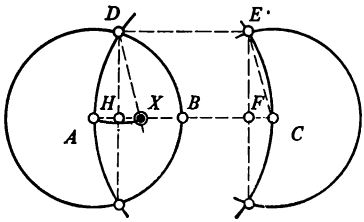

Fig. 13

$(A, |AC|)$, $(C, |AC|)$, and (C, a) which intersect at the points D and E. If we now describe the circles (D, a) and $(C, |DE|)$, then at their intersection we obtain the point X for which $|AX| = \frac{1}{n} |AB|$ (Fig. 13).

Proof. The point X lies on the straight line AB, since $[AC]$ is parallel to $[DE]$ and $[XC]$ is parallel to $[DE]$ (the figure $CEDX$ is a parallelogram). From the similarity of the isosceles triangles ACD and AXD we get

$$|AX| = \frac{1}{n} |AB|.$$

Now we give one more method of construction suggested by A. S. Smogorzhevskii [4]. This construction differs from the preceding constructions in that the required $1/n$th part of the segment AB does not lie on the given segment.

Construction (3d method). We construct $|AC| = n \cdot |AB|$ (Problem 2) and draw the circles $(A, |AC|)$ and $(B, |AC|)$. We take one of their intersection points D and draw the circle $(D, |AB|)$ which intersects the circles at the points E and H. The segment EH is the required one (Fig. 14).

Proof. The triangles ADE and BDH are congruent (three sides are equal) consequently, $\widehat{ADB} = \widehat{EDH}$. From the

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* For the definition of the angle of intersection of two curves see Sec. 8, p. 67.

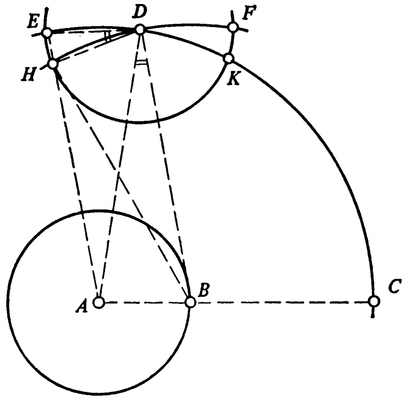

Fig. 14

similarity of the isosceles triangles ADB and EDH it follows that

$$|EH|/|ED| = |AB|/|AD|,$$

or

$$|EH|/a = a/na.$$

Finally

$$|EH| = \frac{1}{n}a = \frac{1}{n}|AB|.$$

We note that

$$|EK| = \frac{\sqrt{4n^2-1}}{n}|AB|,$$

$$|HK| = \left(2 - \frac{1}{n^2}\right)|AB|.$$

Problem 10. Construct a segment 2^n times smaller than a given segment AB (divide a segment AB into 2^n equal parts, $n = 1, 2, 3, \ldots$).

Given $[AB]$ and $n \in \mathbf{N}$. Construct $[BX_n]$, $|BX_n| = (1/2^n)|AB|$, where $|AB| = a$.

Construction (1st method). We construct the segment $|AC| = 2|AB|$ (Problem 2). We draw the circle $(C, |AC|)$ and denote by D_1 and D_1' the points where it intersects the circle (A, a). If now we draw the circles $(D_1, |AD_1|)$ and $(D_1', |AD_1'|)$, at their intersection we obtain the point X_1. The segment BX_1 is the required one ($|BX_1| = (1/2)|AB|$) (Fig. 15).

Then we describe the circle $(A, \mid BD_1 \mid)$ and obtain the points D_2 and D_2' where it intersects $(C, \mid AC \mid)$. We draw the circles $(D_2, \mid AD_2 \mid)$ and $(D_2', \mid AD_2' \mid)$ until they meet at the point X_2. The segment BX_2 is the required segment $(\mid BX_2 \mid = (1/2^2) \mid AB \mid)$.

If we further describe the circles $(A, \mid BD_2 \mid)$, $(D_3, \mid AD_3 \mid)$, and $(D_3', \mid AD_3' \mid)$, we get the point X_3. The segment BX_3 is the required segment $(\mid BX_3 \mid = (1/2^3) \mid AB \mid)$ and so on.

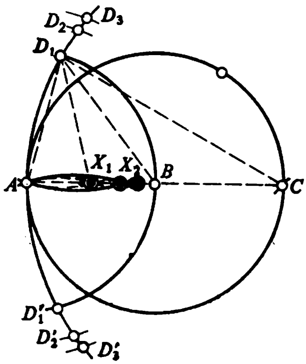

Fig. 15

Proof. From the similarity of the isosceles triangles ACD_1 and AD_1X_1 it follows that

$$\mid AD_1 \mid / \mid AC \mid = \mid AX_1 \mid / \mid AD_1 \mid$$

or

$$a/2a = \mid AX_1 \mid / a.$$

Hence

$$|AX_1| = \frac{1}{2} a = \frac{1}{2} |AB| = |BX_1|.$$

We introduce the notation $\mid BD_k \mid = m_k$, $k = 1, 2, 3, \ldots$ $\ldots, n$. The segment BD_1 is the median of the triangle ACD_1, consequently

$$4 \mid BD_1 \mid^2 = 2 \mid AD_1 \mid^2 + 2 \mid CD_1 \mid^2 - \mid AC \mid^2,$$

or, in another way,

$$4m_1^2 = 2a^2 + 2 \mid AC \mid^2 - \mid AC \mid^2$$
$$= 2a^2 + \mid AC \mid^2 = 2a^2 + 4a^2.$$

This means that

$$m_1^2 = |BD_1|^2 = \frac{1+2}{2} a^2 = \frac{3}{2} a^2.$$

From the similarity of the isosceles triangles ACD_2 and AD_2X_2 we get

$$|AD_2|/|AC| = |AX_2|/|AD_2|$$

and taking into account that $|AD_2| = |BD_1| = m_1$ and $|AC| = 2a$, we have

$$|AX_2| = \frac{3}{4} a \quad \text{or} \quad |BX_2| = \frac{1}{4} a = \frac{1}{2^2} |AB|.$$

Similarly we find

$$m_2^2 = \frac{1+2+2^2}{2^2} a^2 \text{ and } |BX_3| = \frac{1}{2^3} |AB|,$$

and so on. In general

$$m_{k-1}^2 = \frac{1+2+2^2+\ldots+2^{k-1}}{2^{k-1}} a^2 \text{ and } |BX_k| = \frac{1}{2^k} |AB|.$$

The point X_n lies on the segment AB. To divide the segment AB into 2^n equal parts, it is necessary to extend the segment BX_n 2, 3, . . ., $2^n - 1$ times (Problem 2). The constructed points divide the segment AB into 2^n equal parts.

Construction (2nd method). We construct the segment $|AC| = 2|AB|$ (Problem 2). To this end on the circle (B, a) we construct the point C which is diametrically opposite to the point A ($|AE| = |EH| = |HC| = a$). We describe the circles $(A, |AC|)$ and $(C, |CE|)$ and denote the points of their intersection by D_1 and D_1' (Fig. 16). The required point X_1 is to be found at the intersection of the circles $(D_1, |CD_1|)$ and $(D_1', |CD_1'|)$. Obviously, $|BX_1| = (1/2)|AB|$.

We describe the circle $(C, |BD_1|)$ which intersects the circle $(A, |AC|)$ at the points D_2 and D_2'. Then, if we describe the circles $(D_2, |CD_2|)$ and $(D_2', |CD_2'|)$, at their intersection we find the required point X_2. The segment BX_2 is the required segment ($|BX_2| = (1/2^2)|AB|$).

Similarly, describing the circles $(C, |BD_2|)$, $(D_3, |CD_3|)$ and $(D_3', |CD_3'|)$ we construct the point X_3. The segment BX_3 is the required segment ($|BX_3| = (1/2^3)|AB|$) and so on.

Proof. From the similarity of the isosceles triangles ACD_1 and CD_1X_1 we get

$$|CX_1|/|CD_1| = |CD_1|/|AC|.$$

Taking into account that $|CD_1| = |CE| = \sqrt{3}a$, we find $|CX_1| = 3a/2$, which means that $|BX_1| = (1/2)|AB|$.

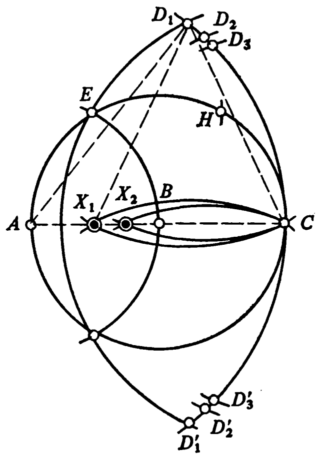

Fig. 16

Let us denote $|BD_k| = m_k$, where $k = 1, 2, \ldots, n$. The segment BD_1 is the median of the triangle ACD_1, consequently

$$\begin{aligned} 4|BD_1|^2 = 4m_1^2 &= 2|AD_1|^2 + 2|CD_1|^2 - |AC|^2 \\ &= 2|AC|^2 + 2|CE|^2 - |AC|^2 \\ &= 4a^2 + 2\cdot 3a^2 \end{aligned}$$

or

$$m_1^2 = \left(1 + \frac{3}{2}\right)a^2.$$

From the similarity of the triangles ACD_2 and AD_2X_2 we have

$$|CX_2|/|CD_2| = |CD_2|/|AC|.$$

Noting that $|CD_2| = |BD_1| = m_1$ and $|AC| = 2|AB| = 2a$, we get

$$|CX_2| = \frac{|CD_2|^2}{|AC|} = \frac{m_1^2}{2a} = \frac{5}{2^2}a.$$

It follows that

$$|BX_2| = \frac{1}{2^2}\,|AB|.$$

In a completely similar way we will prove that

$$m_2^2 = |BD_2|^2 = \frac{9}{4}a^2, \quad |CX_3| = \frac{9}{8}a,$$

$$\text{and} \quad |BX_3| = \frac{1}{2^3}\,|AB|,$$

etc. In general

$$m_{k-1}^2 = |BD_{k-1}|^2 = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{k-2}} + \frac{3}{2^{k-1}}\right) a^2$$

and

$$|BX_k| = \frac{1}{2^k}\,a = \frac{1}{2^k}\,|AB|.$$

If, using the first method of construction for large values of k ($k \leqslant n$) we can not clearly define the point X_k (the

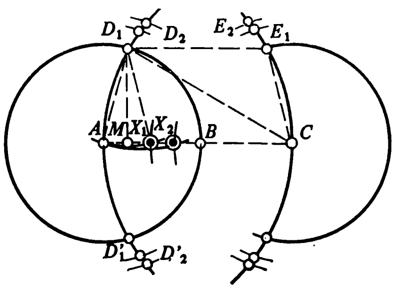

Fig. 17

arcs of the circles which define this point almost coincide), then it is possible to solve the problem as follows.

Construction (3d method). We construct the segment $|AC| = 2\,|AB|$ (Problem 2). We describe the circles $(A, |AC|)$, $(C, |AC|)$, and (C, a) which intersect at the points D_1 and E_1 (Fig. 17). At the intersection of the circles $(D_1, |AD_1|)$ and $(C_1, |D_1E_1|)$ we obtain the point X_1. The segment BX_1 is the required segment ($|BX_1| = (1/2)\,|AB|$).

Further, we construct $[AD_2] \cong [E_2C] \cong [BD_1]$ towards which we describe the circles $(A, |BD_1|)$ and $(C, |BD_1|)$.

We draw the circles $(D_2, |AD_2|)$ and $(C, |D_2E_2|)$ until they meet at the point X_2. The segment BX_2 is the required one ($|BX_2| = (1/2^2)\,|AB|$) and so on.

Proof. Let us set $|BD_k| = m_k$, where $k = 1, 2, \ldots, n$. The point X_1 lies on the straight line AC, since $[AC]$ is parallel to $[D_1E_1]$ (the figure AD_1E_1C is a trapezoid) and $[CX_1]$ is parallel to $[D_1E_1]$ (the figure $X_1D_1E_1C$ is a parallelogram), this means that $[X_1C]$ is parallel to $[AC]$. Similarly, it can be established that the points $X_2, X_3, \ldots$ lie on the straight line AC.

The figure AD_1E_1C is an isosceles trapezoid ($|AD_1| = |CE_1|$), threfore we can write $|AM| = |MX_1|$, where $[MD_1]\perp[AC]$ and $|X_1C| = |D_1E_1|$. It follows that

$$|AD_1| = |D_1X_1| = [AD_1'] = [D_1'X_1] = a.$$

Similarly, it is easy to establish that

$$|AD_2| = |D_2X_2| = |AD_2'| = |D_2'X_2| = m_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|AD_k| = |D_kX_k| = |AD_k'| = |D_k'X_k| = m_{k-1}.$$

Now if we repeat word for word the proof corresponding to the first construction method of the given problem, we obtain

$$|BX_1| = \frac{1}{2}|AB|, \quad |BX_2| = \frac{1}{2^2}|AB|,$$
$$\ldots, |BX_k| = \frac{1}{2^k}|AB|, \ldots$$

Problem 11. Construct a segment 3^n times as great as the segment AA_0 ($n = 1, 2, 3, \ldots$).

Given $[AA_0]$ and $n \in \mathrm{N}$. Construct $[AA_n]$, $|AA_n| = 3^n|AA_0|$.

Construction. We describe the circles $(A, |AA_0|)$ and $(A_0, |AA_0|)$ and denote by E and E' their intersection points (Fig. 18). We draw the circles $(E, |AA_0|)$ and $(E', |AA_0|)$ which will intersect the circle $(A_0, |AA_0|)$ at the points C and C'. Let the point A_1 be the point of intersection of the circles $(C, |AC|)$ and $(C', |AC'|)$. The segment AA_1 is the required segment ($|AA_1| = 3\,|AA_0|$).

The constructions given above for the segment AA_0 may be applied to the segment AA_1. As a result we find the segment AA_2 such that $|AA_2| = 3^2\,|AA_0|$, etc.

Proof. In the equilateral triangle ACC', $|AC| = |AC'| = |CC'| = \sqrt{3}\,|AA_0|$, $h_{\Delta ACC'} = (\sqrt{3}/2)\,|AC| =$

$= (3/2)\ |AA_0|$. Obviously, $|AA_1| = 2h_{\triangle ACC'} = 3\ |AA_0|$. In a similar way we prove that $|AA_2| = 3^2\ |AA_0|$, etc.

Problem 12. Divide the segment AB into three equal parts.

Given $[AB]$. Construct $[AX] \cong [XY] \cong [YB]$, $X \in [AB]$ and $Y \in [AB]$.

Let us consider an elegant method of construction put forward by L. Mascheroni [9].

Construction. We construct the points C and D on the straight line AB so that $|CA| = |AB| = |BD|$ (Prob-

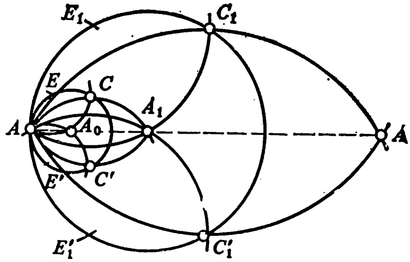

Fig. 18

lem 2). We describe the circles $(C, |CB|)$, $(C, |CD|)$, $(D, |AD|)$, and $(D, |CD|)$, whose intersection points we denote by E, E_1, F, and F_1 (Fig. 19). At the intersections of the circles $(E, |CE|)$ and $(E_1, |CE_1|)$, as well as $(F, |DF|)$ and $(F_1, |DF_1|)$ we find the required points X and Y, which divide the segment AB into three equal parts.

Proof. It follows from the similarity of the isosceles triangles CEX and CDE that

$$|CX|/|CE| = |CE|/|DC|.$$

Taking into account that $|CE| = 2\ |AB|$ and $|CD| = 3\ |AB|$, we obtain $|CX| = (4/3)\ |AB|$, therefore

$$|AX| = \frac{1}{3}\ |AB|.$$

Problem 13. Construct the centre of a given circle.

Construction. On the circumference of the given circle we take a point A and describe the circle (A, d) of an arbitrary radius d. At their intersection we obtain the points B and D. On the circumference of (A, d) we define the point C

diametrically opposite to B. We further draw the circles $(C, |CD|)$ and $(A, |CD|)$ and denote by E the point of their intersection. Finally, we describe the circle $(E, |CD|)$, which intersects the circle (A, d) at the point M.

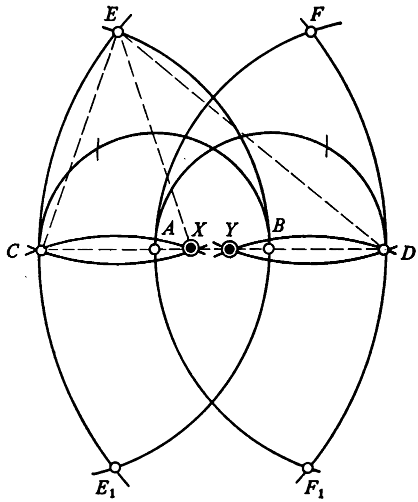

Fig. 19

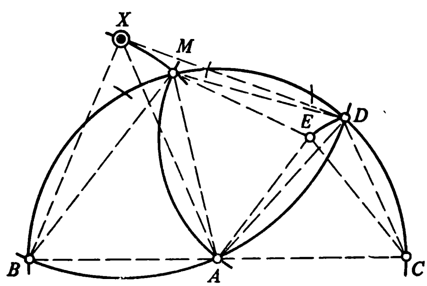

Fig. 20

The segment BM equals the radius of the initial circle, while the circles $(B, |BM|)$ and $(A, |BM|)$ define its centre (Fig. 20).

Proof. The isosceles triangles ACE and AEM are congruent, therefore $\widehat{EAM} = \widehat{ACE}$.

On the one hand, $\widehat{BAE} = \widehat{ACE} + \widehat{AEC}$ ($\angle BAE$ is an exterior angle of the triangle ACE) and, on the other hand, $\widehat{BAE} = \widehat{BAM} + \widehat{EAM}$. Hence $\widehat{BAM} = \widehat{AEC}$.

Thus, the isosceles triangles ABM and ACE are similar, therefore

$$|BM|/|AB| = |AC|/|CE|$$

or

$$|BX|/|AB| = |AC|/|CD|.$$

It follows from the latter that the isosceles triangles ABX and ACD are similar, which means that

$$\widehat{BAX} = \widehat{ACD} = \frac{1}{2}\widehat{BAD} = \widehat{DAX};$$

the latter two equalities follow from the fact that

$$\widehat{BAD} = \widehat{ADC} + \widehat{ACD} = 2\widehat{ACD} = 2\widehat{BAX}.$$

On the basis of the equality of the angles BAX and DAX we conclude that the isosceles triangles ABX and ADX are congruent, therefore

$$|BX| = |AX| = |DX|.$$

The point X is the required centre of the circle.

Note. It is easy to show that the segment $d = |AB|$ should be greater than half the radius of the given circle, otherwise the circles $(C, |CD|)$ and $(A, |CD|)$ do not intersect.

In conclusion to this chapter we give without proof the solution of the following Mascheroni problem [9].

Problem 14. Construct the segment $\frac{1}{2}\sqrt{n}$ as great as the segment AB, where $|AB| = 1$, $n = 1, 2, \ldots, 25$.

Given $[AB]$. Construct $\frac{1}{2}\sqrt{n}\,|AB|$, $n = 1, 2, \ldots, 25$.

Construction. We describe the circle $(A, |AB|)$ and mark the point B on it. We construct the point E diametrically opposite to B ($|BC| = |CD| = |DE| = 1$). We describe the circles $(B, |BD|)$ and $(E, |EC|)$ and let F and F_1 be their intersection points. We describe the circles $(B, |AF|)$ and $(E, |AF|)$ which intersect the circle $(A, |AB|)$ at the points H and H_1, the circle $(B, |BD|)$ at the points N and N_1, and the circle $(E, |EC|)$ at the points M and M_1. We

describe the circles $(E, |AE|)$ and $(B, |AB|)$ and mark the points P and P_1, Q and Q_1 where they intersect the circles $(B, |AF|)$ and $(E, |AF|)$, respectively (Fig. 21). The circles $(P, |BP|)$ and $(P_1, |BP_1|)$ intersect each other at the point R and the circle $(A, |AB|)$ at the points S and S_1. Exactly in the same way the circles $(Q, |EQ|)$ and $(Q_1,$

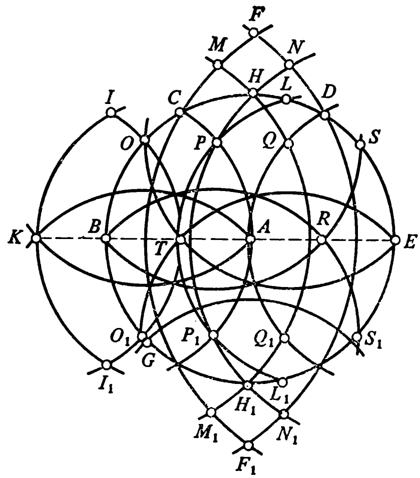

Fig. 21

$|EQ_1|)$ intersect each other at the point T and the circle $(A, |AB|)$ at the points O and O_1. We draw the circles $(R, |AB|)$, and $(F_1, |AB|)$ and denote the points where they intersect the circle $(A, |AB|)$, by L, L_1, and G. Now we describe the circles $(O, |AO|)$ and $(O_1, |AO_1|)$ which intersect at the point K. Finally, we draw the circles $(K, |AB|)$ and $(T, |AB|)$ and mark the points I and I_1 of their intersection. Then

$$|AT| = \frac{1}{2}\sqrt{1}, \quad |PT| = \frac{1}{2}\sqrt{2}, \quad |DR| = \frac{1}{2}\sqrt{3},$$

$$|AB| = \frac{1}{2}\sqrt{4}, \quad |HT| = \frac{1}{2}\sqrt{5}, \quad |AM| = \frac{1}{2}\sqrt{6},$$

$$|QQ_1| = \frac{1}{2}\sqrt{7}, \quad |AF| = \frac{1}{2}\sqrt{8}, \quad |BR| = \frac{1}{2}\sqrt{9},$$

$$|BL| = \frac{1}{2}\sqrt{10}, \quad |PS_1| = \frac{1}{2}\sqrt{11}, \quad |BD| = \frac{1}{2}\sqrt{12},$$

$$|HK| = \frac{1}{2}\sqrt{13}, \quad |BS| = \frac{1}{2}\sqrt{14}, \quad |LL_1| = \frac{1}{2}\sqrt{15},$$

$$|BE| = \frac{1}{2}\sqrt{16}, \quad |FK| = \frac{1}{2}\sqrt{17}, \quad |KN| = \frac{1}{2}\sqrt{18},$$

$$|KD| = \frac{1}{2}\sqrt{19}, \quad |FG| = \frac{1}{2}\sqrt{20}, \quad |I_1D| = \frac{1}{2}\sqrt{21},$$

$$|KS| = \frac{1}{2}\sqrt{22}, \quad |MM_1| = \frac{1}{2}\sqrt{23}, \quad |MN_1| = \frac{1}{2}\sqrt{24},$$

$$|KE| = \frac{1}{2}\sqrt{25} = \frac{1}{2}\sqrt{25}\,|AB|.$$

Sec. 3. Inversion and Its Principal Properties

At the end of the 19th century A. Adler applied the principle of inversion to the theory of geometrical constructions

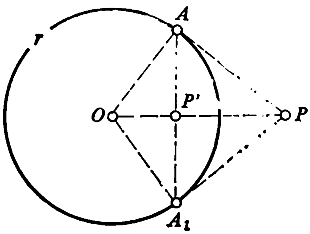

Fig. 22

by compasses alone. With the help of this principle he established a general method of solving construction problems in the geometry of compasses.

In this section we give the definition of inversion and dwell briefly on its principal properties, which are used in our future discussions.

Let a circle (O, r) and a point P other than O be given in the plane of drawing (Fig. 22)*.

On the ray OP we take a point P' so that the product of the segments OP and OP' be equal to the square of the radius of the given circle, i.e.

$$|OP|\cdot|OP'| = r^2. \tag{1}$$

Such point P' is called the *inverse* of the point P with respect to the circle (O, r). The circle (O, r) is called the

* Let us agree to put the radius r of the circle at the break of its arc (see Fig. 22).

circle of inversion, its centre O is the *centre of inversion*, and the quantity r^2 is the *power of inversion*.

If the point P' is inverse of the point P, then, obviously, the point P is inverse of the point P'.

The correspondence between inverse points or, in other words, the transformation is called *inversion** if each point P of some figure corresponds to the inverse point P'.

From the definition of inversion it follows that for each point P in the plane there is always unique point P' in the same plane, and if $|OP| > r$, then $|OP'| < r$. The exception is the centre of inversion O. No point in the plane can be inverse of O, which follows immediately from equality (1)**.

Let (AP) and (A_1P) be tangents to the circle of inversion (O, r) drawn from the point P outside the circle (Fig. 22). Then the intersection point P' of the straight lines AA_1 and OP is the inverse of the point P. Indeed, in the right-angled triangle OAP (AP' is the height)

$$|OP| \cdot |OP'| = |OA|^2 = r^2.$$

Let the point P move along some curve l, then its inverse point P' will also describe some curve l'. The curves l and l' are called *mutually inverse.*

Lemma. *If the points P' and Q' are the inverse of the points P and Q with respect to the circle (O, r), then*

$$\angle OP'Q' \cong \angle OQP \text{ and } \angle OQ'P' \cong \angle OPQ.$$

Proof. From the equalities $|OP| \cdot |OP'| = |OQ| \times |OQ'| = r^2$ or $|OP|/|OQ| = |OQ'|/|OP'|$ it follows that the triangles $OQ'P'$ and OQP are similar (Fig. 23). This proves the lemma.

From the definition of inversion three theorems immediately follow.

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* Let us put $|OA| = r = 1$, $|OP| = R$, and $|OP'| = R'$, equality (1) in this case can be written as $R = 1/R'$, i.e. the distances of the inverse points P and P' from the centre of inversion O are reciprocal numbers. Inversion (Latin *inversio*) literally means turning over, changing places. Inversion can also be named the *reciprocal radii transformation.*

** In the theory of conformal mappings the centre O is associated with an 'infinitely distant point' in the plane because $|OP| \to \infty$ when $|OP'| \to 0$. In general, inversion transforms the points lying in the circle (O, r) into the points lying outside the circle, and vice versa.

Theorem 1. *If two curves intersect at the point P, then the curves inverse of them intersect at the point P' which is the inverse of the point P.*

Theorem 2. *A straight line passing through the centre of inversion O is inverse to itself.*

Theorem 3. *The curve which is inverse to a given straight line AB, not passing through the centre of inversion, is*

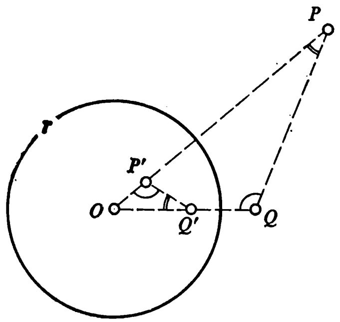

Fig. 23

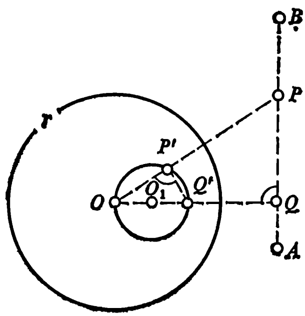

Fig. 24

the circle $(O_1 \mid OO_1 \mid)$, which passes through the centre of inversion O, (OO_1) being always perpendicular to (AB).

Proof. Let Q be the foot of the perpendicular dropped from the centre of inversion O to the given straight line. Let us denote by Q' the point inverse of Q. We take an arbitrary point P on the given straight line and denote the point inverse of it by P' (Fig. 24).

On the basis of the lemma we can write

$$\widehat{OP'Q'} = \widehat{OQP} = 90^\circ.$$

Consequently, when the point P moves along the straight line AB, the inverse P' describes a circle with the segment OQ' as its diameter.

Since the circle $(O_1, \mid OO_1 \mid)$ and the given straight line AB are mutually inverse, the converse proposition also holds, namely, the circle passing through the centre of inversion is the inverse of the straight line.

Theorem 4. *The curve which is inverse to the given circle (O_1, R), not passing through the centre of inversion, is also a circle. Here the centre of inversion is the centre of similitude of these circles.*

Proof. Let the line OO_1 joining the centres of the circle of inversion (O, r) and the given circle (O_1, R) intersects the latter at the points A and B. Let us denote by A' and B' the inverses of the points A and B. Let us take an arbitrary point

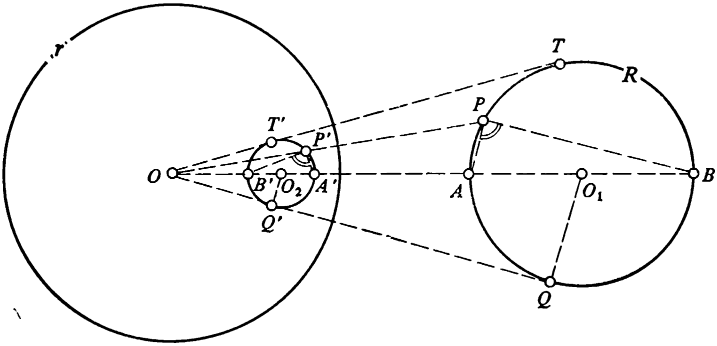

Fig. 25

P on the circle (O_1, R) and denote the inverse of it by P' (Fig. 25). Applying the lemma, we obtain

$$\angle OA'P' \cong \angle OPA \text{ and } \angle OB'P' \cong \angle OPB,$$

whence

$$\widehat{OB'P'} - \widehat{OA'P'} = \widehat{OPB} - \widehat{OPA}.$$

In the triangles $A'P'B'$ and APB

$$\widehat{AP'B'} = \widehat{OB'P'} - \widehat{OA'P'} \text{ and } \widehat{APB} = \widehat{OPB} - \widehat{OPA} = 90°.$$

Taking into account the preceding equation, we get

$$\widehat{A'P'B'} = \widehat{APB} = 90°.$$

Now let the point P move along the given circle (O_1, R), then the inverse of it, P', describes the circle $(O_2, \mid O_2P' \mid)$, which has the segment $A'B'$ as its diameter. The theorem has been proved.

If QQ' and TT' are the common outside tangents of the given circle (O_1, R) and the circle $(O_2 \mid O_2P' \mid)$ inverse of it, then the points of tangency Q, Q' and T, T' are always mutually inverse. The perpendicular at the point Q' to the tangent QQ' intersects the line of centres OO_1 at the point O_2, which is the centre of the circle inverse to the given one.

From the right triangles OO_1Q and OO_2Q' we can write

$$\frac{|OO_1|}{|OO_2|}=\frac{|OQ|}{|OQ'|}=k,$$

where k is the homothetic ratio of the circles $(O_2, \mid O_2P' \mid)$ and (O_1, R). Hence,

$$\mathbf{H}_O^k(\smile Q'B'T')=\smile QAT \text{ and } \mathbf{H}_O^k(\smile Q'A'T')=\smile QBT,$$

where $\mathbf{H}_O^k$ is the homothetic (similitude) transformation with the homothetic centre O and the ratio k.

It should be noted that $\smile Q'B'T'$ is inverse of $\smile QBT$ and $\smile Q'A'T'$ is inverse of $\smile QAT$.

Sec. 4. The Application of the Inversion Method to the Geometry of Compasses

The application of the inversion method to the solution of geometrical construction problems by means of compasses

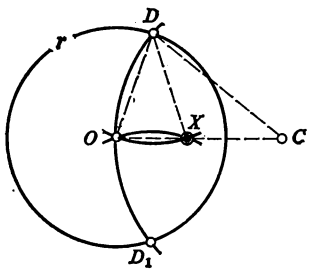

Fig. 26

alone yields a general approach to the solution of construction problems in the geometry of compasses.

The constructions of Mohr and Mascheroni, although extremely elegant, nevertheless are in most cases performed by such artificial means that the question arises how they could find them.

Problem 15. Construct a point X inverse of the given point C with respect to the circle of inversion (O, r).

Given (O, r) and the point C. Construct $X \in [OC)$, $|OX|\cdot|OC| = r^2$.

Construction in the case of $|OC| > r/2$ (Fig. 26). We draw the circle $(C, |OC|)$ and denote by D and D_1 the points where it intersects the circle of inversion (O, r). If now the

circles (D, $|OD|$) and (D_1, $|OD_1|$) are drawn, at the intersection we obtain the required point X.

Proof. From the similarity of the isosceles triangles CDO and DOX we find

$$|OC|/|OD| = |OD|/|OX|,$$

or

$$|OC|\cdot|OX| = |OD|^2 = r^2.$$

Note. It is easy to see that the construction coincides with that given in Problem 9 (1st method) if the point C is considered to be given instead of constructing the segment $|AC| = n|AB|$. Solving Problem 15, we may also em-

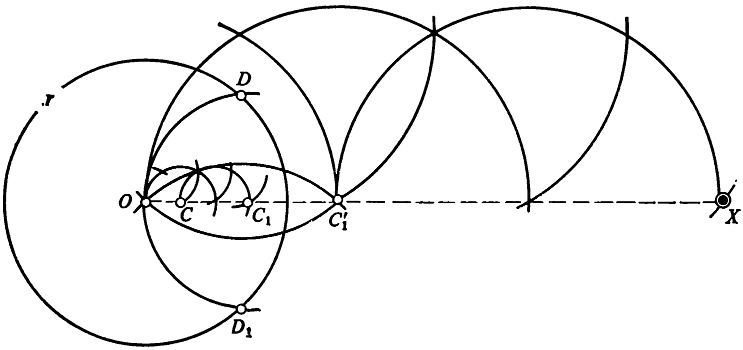

Fig. 27

ploy the note to the first method of solution to Problem 9. It is obvious that here we can also use the second method of solving Problem 9 (in this case the segment $|AC| = n|AB|$ should not be constructed too).

Construction in the case of $|OC| \leqslant r/2$ (Fig. 27). The circle (C, $|OC|$) will not intersect the circle of inversion, therefore we construct the segment $|OC_1| = n|OC|$, taking a natural number n such that $|OC_1| > r/2$ (Problem 2). We find the point C_1' inverse of the point C_1 (1st method of construction) and construct the segment $|OX| = n|OC_1'|$. The point X is inverse of the given point C.

Proof. Substituting $|OC_1| = n|OC|$ and $|OC_1'| = |OX|/n$ in the equation $|OC_1|\cdot|OC_1'| = r^2$, we obtain

$$|OC_1|\cdot|OC_1'| = n|OC|\cdot\frac{|OX|}{n} = |OC|\cdot|OX| = r^2.$$

Note. The construction given above is possible if the point C is not the centre of inversion.

Problem 16. Given the circle of inversion (O, r) and the straight line AB which does not pass through the centre of inversion. Construct the circle which is the inverse of the given straight line.

Given (O, r) and (AB) (O does not belong to (AB)). Construct $(O_1', |OO_1'|)$ inverse of (AB).

Construction. We construct O_1 symmetric to the centre of inversion O with respect to the straight line AB (Prob-

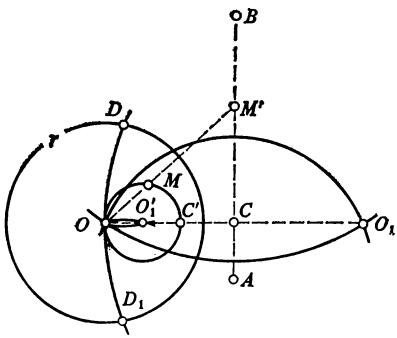

Fig. 28

lem 1). We find the point O_1', inverse of the point O_1 (Problem 15). The circle $(O_1', |OO_1'|)$ is inverse of the given straight line AB (Fig. 28).

Proof. Let C and C' be the intersection points of the straight line OO_1 with the given straight line AB and the circle $(O_1', |OO_1'|)$, respectively.

From the given construction it follows that

$$|OO_1|\cdot|OO_1'| = r^2, \quad |OO_1| = 2\,|OC|,$$
$$|OC'| = 2\,|OO_1'|, \quad (OC) \perp (AB).$$

Hence

$$|OO_1|\cdot|OO_1'| = 2\,|OC|\cdot\frac{|OC'|}{2} = |OC|\cdot|OC'| = r^2.$$

By virtue of Theorem 3 the circle $(O_1', |OO_1'|)$ is the inverse of the straight line AB.

Note. If a straight line passes through the centre of inversion, then it is inverse to itself (Theorem 2).

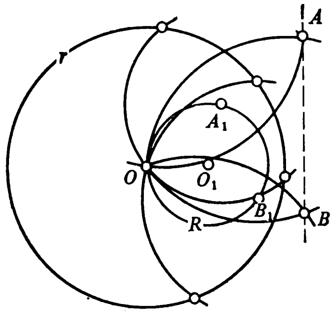

Fig. 29

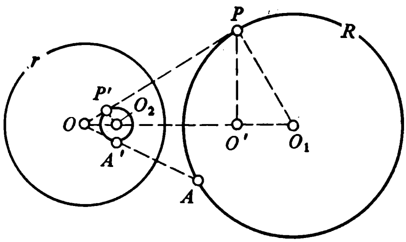

Fig. 30

Problem 17. Construct the straight line AB which is the inverse of the given circle (O_1, R) passing through the centre of inversion O.

Construction. If the given circle intersects the circle of inversion at the points A and B, then the straight line AB is the inverse of this circle. Otherwise,we take the points A_1 and B_1 on the given circle (Fig. 29) and construct their inverses (Problem 15). The straight line AB is the inverse of the given circle (O_1, R).

By varying the position of the points A_1 and B_1 on the given circle (or using Problem 5), it is possible to construct as many points of this straight line as required.

The validity of the construction follows from Theorem 3.

Problem 18. The circle (O_1, R) does not pass through the centre of inversion O. Construct the circle which is the inverse of given one.

Construction. We take the circle (O_1, R) as the circle of inversion and construct the point O' inverse of the point O

(Problem 15). Then we construct the point O_2 inverse of the point O' with respect to the circle of inversion (O, r). The point O_2 is the centre of the required circle (Fig. 30).

We take any arbitrary point A on the circle (O_1, R) and find the point A' inverse of it. The circle $(O_2, |O_2A'|)$ is inverse of the given circle (O_1, R).

Proof. Let PP' be the common outside tangent to the circles (O_1, R) and $(O_2, |O_2A'|)$ and let (PO') be perpendicular to (OO_1).

From the similarity of the right-angled triangles OPO' and $OP'O_2$, we can write

$$|OO_2|/|OP'| = |OP|/|OO'|$$

or, in other words,

$$|OO_2|\cdot|OO'| = |OP|\cdot|OP'| = r^2,$$

since the points P and P' are mutually inverse. It follows from the last equation that the points O_2 and O' are inverse with respect to the circle of inversion (O, r).

In the right-angled triangle OO_1P the segment $O'P$ is the height, therefore,

$$|O_1O|\cdot|O_1O'| = |O_1P|^2 = R^2.$$

Thus, the point O' is the inverse of the point O with respect to the circle (O_1, R) if the latter is taken as the circle of inversion.

The point O is given. When constructing, the point O' was found first, followed by the point O_2, which is the centre of the required circle.

Note. Employing more complex calculations, we can prove that the given construction remains valid when the centre of inversion O is inside the given circle (O_1, R)*.

* * *

It was shown in Problems 15-18 how to construct figures which are the inverse of a point, a straight line, and a circle, using compasses alone. Now we can consider a general method of solving geometrical construction problems with compasses alone.

Each construction carried out by compasses and a ruler gives in the plane of the drawing a figure Φ consisting of

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* **The proof is given in Appendix 2.**

circles, straight lines, and separate points. The figure Φ' inverse of Φ with respect to the circle (O, r), which is taken as the circle of inversion, whose centre O is lying neither on straight lines nor on circles of the figure Φ, consists *only of points and circles.* Using Problems 15-18 we see that each of these points and straight lines can be constructed by compasses alone.

Now let us assume that we are confronted with a construction problem solvable by a ruler and compasses having only compasses at our disposal.

Let us imagine that this problem has been solved by compasses and a ruler and as a result a certain figure Φ has been obtained which consists of points, straight lines, and circles. We will construct this figure by drawing a finite number of straight lines and circles in a definite order.

We take the most suitable circle of inversion (O, r) and construct the figure Φ' which is the inverse of the figure Φ (Problems 15-18). The figure Φ' consists only of points and circles provided, of course, the circle of inversion has been chosen such that its centre lies neither on straight lines nor on circles of the figure Φ.

If now we construct the inverse of the figure which is taken as the result in the figure Φ', then we arrive at the required result. We note here that we should carry out the construction of the figure Φ' in the same order that we used while constructing the figure Φ by compasses and a ruler.

By means of the method described above it is possible to solve by compasses alone each construction problem solvable by compasses and a ruler. Thus the basic result obtained by Mohr-Mascheroni has been proved once again with the help of the method of inversion.

The five simplest problems mentioned at the end of Sec. 1 can also be solved by the general method.

We take the solution of Problem 7 as an illustration of the general method of solving construction problems by compasses alone: construct the intersection point of two straight lines AB and CD, each of which is given by two points.

We take an arbitrary circle (O, r) with the centre O not lying on the given straight lines and regard it as the circle of inversion. We construct the circles which are the inverses of the given straight lines and mark the point of intersection, X' (Problem 16). We construct the point X inverse of the

point X' (Problem 15). The point X is the required intersection point of the given straight lines AB and CD.

Here the figure Φ consists of two straight lines AB and CD (more exactly, it consists of four given points A, B, C, and D, through which we mentally draw the given lines). The figure Φ' consists of two circles, inverses of the straight lines AB and CD. The image taken as the result in the figure Φ' will be the point X'. The point X inverse of the point X' is the required result, that is, the intersection point of the given straight lines.

Exactly in the same way it is possible to solve Problem 6 (the fourth simplest problem): to construct the points of

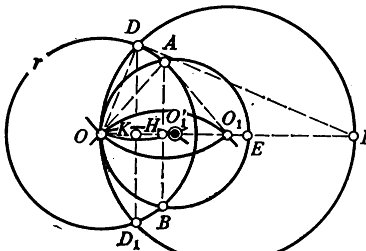

Fig. 31

intersection of a straight line and a circle. If the straight line does not pass through the centre of the circle, then the circle should be regarded as the circle of inversion. Then the solution of the problem will be considerably simplified.

The validity of these constructions follows immediately from Theorem 1.

Problem 19. Find the centre of a given circle.

Construction. We take a point O on the given circle and describe a circle (O, r) of an arbitrary radius r which intersects the given circle at the points A and B. We take the circle (O, r) as the circle of inversion and construct the centre of the circle which is the inverse of the straight line AB (Problem 16). To carry out the latter construction, we draw the circles $(A, |OA|)$ and $(B, |OB|)$ until they meet at the point O_1. We describe the circle $(O_1, |OO_1|)$ and mark the points D and D_1 where it intersects the circle of inversion. The circles $(D, |OD|)$ and $(D_1, |OD_1|)$ define the required centre of the original circle (Fig. 31).

Proof. The points A and B are inverses of each other, since they lie on the circle of inversion. Thus, the given circle and the straight line AB are mutually inverse figures.

In Problem 16 it is shown that the point O_1' is the required centre of the given circle, which in this case is the inverse of the straight line AB.

We should like to draw the reader's attention to the simplicity and elegance of the solution of the last problem. In order to find the centre of the circle, six circles have been drawn*. This construction is simpler and more exact than the usual construction with a ruler and compasses.

This and also certain of the problems in the geometry of compasses as, for example, Problems 3 and 8 (2nd method) can be given to senior pupils to practice geometry. For this reason, we give the proof of Problem 19 which is not based on the principle of inversion.

Proof. The straight line OO_1 is perpendicular to the chord AB of the circle and passes through its midpoint, therefore the required centre must lie on the straight line OO_1. Let E and F be the intersection points of the straight line OO_1 with the given circle and with the circle $(O_1, |OO_1|)$, respectively. The segment OE is the diameter of the given circle.

Examining the right-angled triangles OAE and ODF in which AH and DK are the heights, respectively, we find

$$|OA|^2 = |OE|\cdot|OH| \text{ and } |OD|^2 = |OF|\cdot|OK|.$$

Taking into account that

$$|OD| = |OA| = r, \quad |OF| = 2|OO_1|,$$

$$|OH| = \frac{1}{2}|OO_1|, \text{ and } |OK| = \frac{1}{2}|OO_1'|,$$

we obtain

$$|OE|\cdot|OH| = |OF|\cdot|OK|$$

or

$$|OE|\cdot\frac{|OO_1|}{2} = 2|OO_1|\frac{|OO_1'|}{2}.$$

Hence

$$|OO_1'| = \frac{|OE|}{2}.$$

* Provided, of course, that the radius r is taken greater than half the radius of the given circle. Otherwise, there will be more circles (see the solution to Problem 15, the second case).

Problem 20. Circumscribe a circle round a given triangle ABC.

Construction. We describe the circle $(A, |AB|)$ and take it as the circle of inversion. We construct the point C' the inverse of the point C (Problem 15) and then the circle $(X, |AX|)$, which is the inverse of the straight line BC' (Prob-

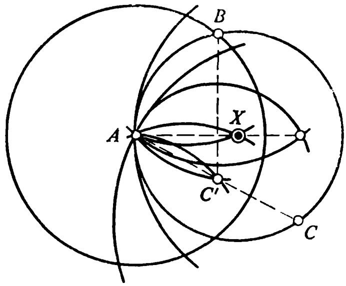

Fig. 32

lem 16). The circle $(X, |AX|)$ is the required circle, circumscribed about the triangle ABC (Fig. 32).

Proof. The point B is its own inverse since it lies on the circle of inversion $(A, |AB|)$. The point C' is the inverse of the point C by construction. Consequently, the circle passing through the given points A, B, and C is the inverse of the straight line BC'. And, as was shown in Problem 16, the point X is the centre of the required circle*.

Note. Now we give the following method of solving Problem 18. We take arbitrary points A, B, and C on the given circle (O_1, R) and construct their inverse A', B', and C'. The circle circumscribed about the triangle $A'B'C'$ is the required circle, inverse of the given one.

* In Problem 16 we constructed the centre of a circle, which was inverse of a given straight line. This construction was used in solving Problems 19 and 20.

2. GEOMETRICAL CONSTRUCTIONS BY COMPASSES ONLY UNDER CONSTRAINTS

In Chapter 1 we investigated constructions by compasses alone which we can now call the *classical geometry of compasses.*

By the theory of geometrical constructions by compasses alone, we mean the free use of the compasses. In other words, no constraints are put on the angle made by the compasses legs when it is possible to draw circles whose radii as large or as small as we please.

It is well known, however, that in practice it is possible to describe circles whose radii are no larger than $R_{\max}$ and no smaller than $R_{\min}$. The length $R_{\max}$ corresponds to the maximum and $R_{\min}$ to the minimum opening of the legs of the given compasses. If we denote by r the radius of a circle which can be described with these compasses, the following inequality

$$R_{\min} \leqslant r \leqslant R_{\max}$$

always holds.

We say that in this case the opening of the legs of the compasses is bounded *from below* by the segment $R_{\min}$ and *from above* by the segment $R_{\max}$.

In this chapter we consider geometrical constructions by compasses alone when certain constraints are imposed on the opening of the legs.

Sec. 5. Constructions by Compasses Only When the Opening of the Legs Is Bounded from Above

In this section we use compasses the opening of whose legs is bounded *only from above* by a certain preassigned value $R_{\max}$. With such compasses it is possible to describe circles whose radii do not exceed this segment. For the sake of brevity, in what follows, we write R instead of $R_{\max}$. If we denote by r the radius of a circle which can be drawn by the given compasses, then always

$$0 < r \leqslant R.$$

Problem 21. Construct a segment 2^n times smaller than a given segment AB (divide a segment AB into $2, 4, 8, \ldots, 2^n$ equal parts).

Construction. It is easy to verify that for the case $|AB| \leqslant \leqslant \frac{R}{2}$* it is possible to use the construction given in Problem 10; the radius of the greatest circle in that construction is equal to $|AC| = 2|AB| \leqslant R$**.

When $|AB| < 2R$ we describe the circles (A, r) and (B, r) of an arbitrary radius r and denote by C and D the points of intersection. Varying the size of r, it is always possible to get $|CD| \leqslant R/2$. Now we bisect the segment CD (Problem 10) and obtain the point X_1. The point X_1 obviously bisects the given segment AB.

Exactly in the same way we construct the point X_2, which bisects the segment AX_1. The segment $|AX_2| = (1/4)|AB| \leqslant \leqslant R/2$. The construction of points $X_4, X_8, \ldots, X_{2n}$ can be reduced to solving Problem 10.

If we increase $|AX_{2n}| = \frac{|AB|}{2^n}$ 2^n times (Problem 2), we divide the segment AB into 2^n equal parts.

The construction for the case $|AB| \geqslant 2R$ will be given in Problem 24.

---

* To compare two given segments AB and CD we describe the circle $(A, |CD|)$. If the point B lies (a) inside this circle, then $|AB| < < |CD|$, (b) on the circumference, then $|AB| = |CD|$, and (c) outside the circle, then $|AB| > |CD|$.

To verify the inequality $|AB| \leqslant R/2$ or the inequality $2|AB| \leqslant \leqslant R$, we must draw the circle (A, R). If the point B lies on the circumference (A, R) or outside it, then $2|AB| > |AB| \geqslant R$; if the point B lies inside the circle, then $|AB| < R$ and therefore the segment $2|AB|$ can be constructed (Problem 2) and compared with the segment R in the manner shown above.

** In the first method of construction in Problem 10 it is necessary to check whether $|AD_n| \leqslant R$ for all $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} |AD_n|^2 = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)a^2$$

$$=\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots\right)a^2$$

$$=\frac{a^2}{1-\frac{1}{2}}=2a^2=2|AB|^2,$$

i.e.

$$|AD_n| < \sqrt{2}\,|AB| < R.$$

Problem 22. (The first basic operation). Construct one or several points on the straight line given by two points A and B*.

Construction. The case when $|AB| < 2R$ is reduced to Problem 5.

Let $|AB| \geqslant 2R$. We describe the circles (B, R) and (A, r), where r is an arbitrary segment smaller than or equal to R. We take a point C on the circumference of the circle

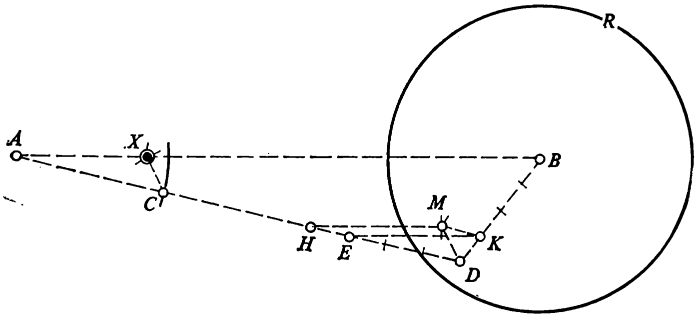

Fig. 33

(A, r) such that it should lie "approximately" on the segment AB (i.e. such that the angle CAB be as small as possible) and construct the segment $|AD| = m|AC|$ (Problem 2, $|AC| = r \leqslant R$). We select the natural number m so that the point D falls within the circle (B, R)**. Varying the position of the point C on the arc (A, r) and, if necessary, varying the size of r, it is always possible to get the point D to lie inside the circle (B, R). At the same time we construct the segments $|AC| = \ldots = |HD| = |AD|/m$ (Fig. 33). Let us take a natural number n such that $2^{n-1} < m \leqslant 2^n$. We construct the segment $|DK| = (1/2^n) \times |BD|$ (Problem 21, here $|BD| < 2R$). We divide the segment DH into 2^n equal parts (Problem 21, $|DH| =$

---

* As we have already noted, we cannot draw a continuous straight line by compasses alone, still less by compasses the opening of whose legs is bounded. However we are able to construct any number of points of this straight line.

** The point D need not lie inside the circle (B, R); it is important that $|BD| < 2R$. The point should lie in the circle $(B, 2R)$ but we cannot describe such a circle by the given compasses.

$r \leqslant R$) and take a segment $|DE| = (m/2^n)\ |DH|$ (in Fig. 33 $m = 3$, $2^{2-1} < 3 < 2^2$, $n = 2$, $|DE| = \frac{3}{4}|DH|$).

We construct a parallelogram $HEKM$, for which purpose it is necessary to draw the circles $(H, |EK|)$ and $(K, |EH|)$. (If at the intersection of these circles the point M is not clearly defined, then in order to construct the point M, we should draw the circle $(E, |EK|)$ and mark off on it the chords $|KP| = |PT|$ equal to the radius $|EK|$. At the intersection of the circles $(H, |EK|)$ and $(P, |TH|)$ we get the point M.)

Finally, if we draw the circles $(A, |HM|)$ and $(C, |DM|)$, they intersect at the required point X, lying on the straight line AB.

Further construction of the points which belong to the given straight line AB reduces to Problem 5 ($|AX| < 2R$).

Proof. From the construction we have

$$\frac{|BD|}{|DK|} = 2^n \text{ and } \frac{|AD|}{|DE|} = \frac{m\,|AC|}{\frac{m}{2^n}\,|AC|} = 2^n.$$

Thus the triangles ADB and EDK are similar (the angle ADB is common). Hence,

$$\angle DEK \cong \angle DAB \text{ and } [EK] \parallel (AB).$$

Since $[HM]$ is parallel to $[EK]$ (the figure $HEKM$ is a parallelogram), we have

$$[HM] \parallel (AB).$$

From the congruence of the triangles ACX and DHM it follows that $[AX]$ is parallel to $[HM]$, i.e. the point X lies on the straight line AB.

The radii of all the circles drawn in this construction do not exceed the segment R.

Note. If $m = 2^n$, i.e. m takes one of the values 2, 4, 8, 16, . . ., the construction of this problem is considerably simplified. In this case the point E coincides with the point H and the point M coincides with the point K. In this case there is no need to divide the segment DH into 2^n equal parts and to construct the parallelogram $EKMH$.

Thus, while varying the size of r, we should always make sure that the number m takes one of the values 2, 4, 8, 16, . . .

Problem 23. Lay off a segment from the point C parallel and equal to a given segment AB.

Construction. If the point C does not lie on the straight line AB, the problem is reduced to the construction of the parallelogram $ABDC$ ($\overrightarrow{AB} \uparrow\uparrow \overrightarrow{CD}$) or $ABCD'$ ($\overrightarrow{AB} \uparrow\downarrow \overrightarrow{CD'}$)*.

Let $|AB| \leqslant R$ and $|AC| \leqslant R$. Suppose that the point C does not lie on the straight line AB. We describe the cir-

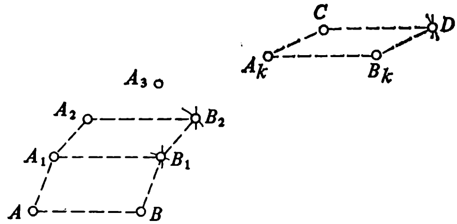

Fig. 34

cles (C, $|AB|$) and (B, $|AC|$) and mark the point of their intersection D. Then CD is the required segment and $ABDC$ is a parallelogram.

If it is necessary to lay off the segment from the point C in the opposite direction ($\overrightarrow{AB} \uparrow\downarrow \overrightarrow{CD}$), then we have to draw the circle (A, $|BC|$) instead of (B, $|AC|$). In case of $|BC| > R$ we cannot describe the circle (A, $|BC|$) by the given compasses. We can obtain the required point, however, if we construct on the circumference of the circle (C, $|AB|$) the point D', diametrically opposite to D. The figure $ABCD'$ is the required parallelogram.

Now let $|AC| > R$ and $|BC| > R$ (Fig. 34). We take an arbitrary set of points $A_1, A_2, \ldots, A_k$ from the point A towards the point C, provided that $|AA_1| \leqslant R$, $|A_1A_2| \leqslant \leqslant R, \ldots, |A_kC| \leqslant R$, and construct the parallelograms ABB_1A_1, $A_1B_1B_2A_2, \ldots, A_{k-1}B_{k-1}B_kA_k$. Then we construct the parallelogram A_kB_kDC (or A_kB_kCD'). The segment CD is the required one. If the point A_i happens to lie on the straight line $A_{i-1}B_{i-1}$, then it will be necessary to take some other point instead of A_i.

This construction also holds when the point C lies on the straight line AB.

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* See Appendix 1.

Now let us consider the case $|AB| > R$. Making use of the solution of Problem 22, we construct points $X_1, X_2, \ldots, X_n$ on the segment AB on condition that $|AX_1| \leqslant R$, $|X_1X_2| \leqslant R, \ldots, |X_nB| \leqslant R$.

We then construct parallelograms $AX_1D_1C, X_1X_2D_2D_1, \ldots, X_{n-1}X_nD_nD_{n-1}, X_nBDD_n$. The segment CD is the required one.

Problem 24. Construct a segment 2^n times smaller than a given segment AB when $|AB| \geqslant 2R$ (divide a segment into 2^n equal parts).

Construction. On the given segment AB we find a point C such that $|AC| \leqslant R$ (Problem 22). We construct the seg-

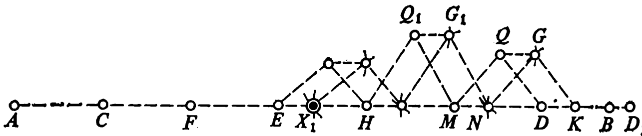

Fig. 35

ment $|AD| = m|AC|$ (Problem 2) taking the natural number m such that $|AD| \leqslant |AB|$ and $|DB| < R$. To this end we lay off the segment AC two, three, etc. times until we approach the point B. If, at the end, the number m turns out to be odd, then we construct the additional segment $|DD_1| = |AC|$. Thus $|AD_1| = (m + 1)|AC|$, $|AB| < |AD_1|$ and $|BD_1| < R$ (in Fig. 35, $m = 6$).

We bisect the segment BD (or BD_1) at the point K (Problem 21, $|BD| < R$).

We denote the midpoint of the segment AD (or AD_1) by E and lay off the segment EX_1, equal and parallel to the segment DK (Problem 23) such that $|AX_1| = |AE| + |DK|$ (or $|AX_1| = |AE| - |DK|$, if the point E is the middle of the segment AD_1). To achieve that, we take the points $Q, Q_1, \ldots$ and construct parallelograms $QDKG$, $MQGN$, Q_1MNG_1, and so on*.

The point X_1 bisects the given segment AB.

After that, we bisect the segment AX_1 and obtain a quarter of the segment AB, etc. If it happens that $|AX_1| < 2R$, we use the construction indicated in Problem 21, otherwise we carry out the construction similar to the above.

* The point X_1 is considered to be constructed if $|AX_1| = |AE| \pm |DK|$ (see Note to Problem 6).

Problem 25. Construct a segment n times greater than the given segment AB in case of $|AB| > R$.

Construction. On the given straight line AB we find a point C such that $|AC| < R$ (Problem 22). We construct the segment $|AD| = m|AC|$ (Problem 2, $|AC| < R$), selecting a number m in such a way that $|AD| \leqslant |AB|$

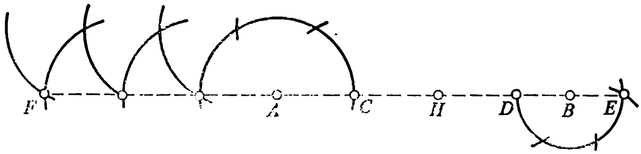

Fig. 36

and $0 \leqslant |DB| < R$. To this end it is necessary to lay off the segment AC two, three, etc. times, until we reach the point B (Fig. 36).

We construct the segment $|DE| = n|DB|$ (Problem 2, $|DB| < R$) so that $\overrightarrow{AD} \uparrow\uparrow \overrightarrow{DE}$ (see Appendix 1), and then the segment $|AF| = (n-1)m|AC|$ with $\overrightarrow{AF} \uparrow\downarrow \overrightarrow{AB}$. The segment $|FE| = n|AB|$ is the required one (in Fig. 36 $m = 3$, $n = 2$).

Proof.

$$\begin{aligned} |FE| &= |FA| + |AD| + |DE| \\ &= (n-1)m|AC| + m|AC| + n|DB| \\ &= nm|AC| + n|DB| = n(m|AC| + DB|) \\ &= n(|AD| + |DB|) = n|AB|. \end{aligned}$$

Note. In order to construct the segment $|AM| = n|AB|$, it is necessary to construct' the segment $|EM| = (n-1)m|AC|$ instead of the segment AF so that $\overrightarrow{EM} \uparrow\uparrow \overrightarrow{AB}$. Then we obtain $|AM| = n|AB|$, i.e. the given and the constructed segments have the same end.

Problem 26. (The second basic operation). Describe a circle of a given radius $|AB|$ with a given point O as centre.

Construction. If $|AB| \leqslant R$, then the circle can be described directly by means of the given compasses with a bounded opening. But if $|AB| > R$, then we cannot draw a

circle in the form of a continuous curve using the given compasses. However, in this case, any number of points can be constructed as close together as desired on the required circle, whose centre and radius are given (Fig. 37).

We construct the segment $a = |AB|/2^n$ (Problems 21 and 24), taking the number n such that $a \leqslant R$. We describe

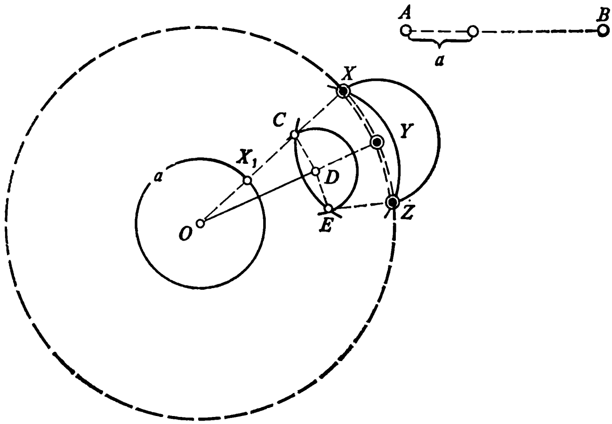

Fig. 37

the circle (O, a), take an arbitrary point X_1 on it, and construct the segment $|OX| = 2^n |OX_1|$ (Problem 2, $|OX_1| = a \leqslant R$). The point X lies on the circle $(O, |AB|)$.

Varying the position of the point X_1 on the circle (O, a), we can construct as many points of the required circle as desired.

When the points X and Y of the required circle are already constructed, and if $|XY| < R$, $|DX| \leqslant R$, we can construct further points on the circle as follows. We describe the circles $(Y, |XY|)$ and $(D, |DX|)$ which have the intersection point Z. The point Z belongs to the given circle. We describe the circles $(D, |DC|)$ and $(Y, |CY|)$ and denote by E the point of their intersection. If we then draw the circles $(Z, |YZ|)$ and $(E, |EY|)$, we obtain one more point on the circle, and so on.

Proof. The segment $|OX| = 2^n a = 2^n |AB|/2^n = |AB|$.

Note. If on the segment AB one more point K is given such that $|AK| \leqslant R$ (or $|BK| \leqslant R$), then the construction can be considerably simplified. We construct the segment $|AD| = m\,|AK|$ (Problem 2, $|AK| \leqslant R$), picking the number m so that $|AD| \leqslant |AB|$ and $0 \leqslant |DB| < R$. To do this, we should lay off the segment AK two, three, etc. times (Fig. 36). If now we describe the circle (O, a),

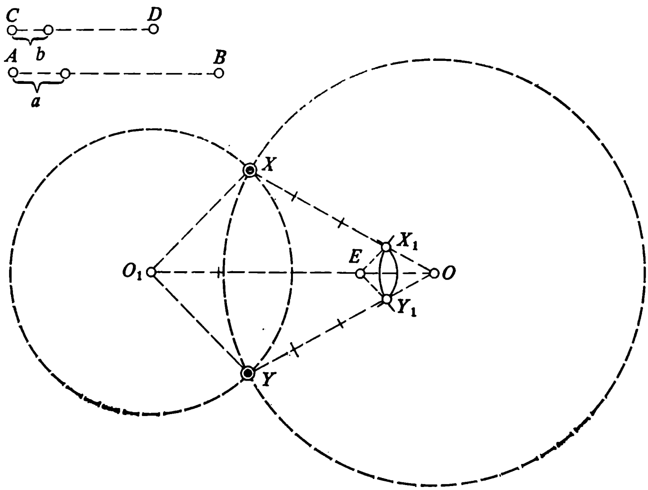

Fig. 38

where $a = |AK|$, take the point X_1 on it, construct the segment $|OC| = m\,|OX_1|$, and, finally, lay off the segment $|CX| = |DB|$* ($\overrightarrow{OC} \uparrow\uparrow \overrightarrow{CX}$), then the point X lies on the given circle (see Fig. 37).

Problem 27. (The third basic operation). Find the points of intersection of two given circles $(O, |AB|)$ and $(O_1, |CD|)$.

Construction. If the radii of both circles are no greater than R, the construction of their intersection points is carried out directly by compasses.

Now suppose that the radius of one or both circles is greater than R.

---

* See Note to Problem 6.

We construct the segments $a = |AB|/2^n$, $b = |CD|/2^n$, and $|OE| = |OO_1|/2^n$ (Problems 21 and 24); we take the number n such that $a \leqslant R$ and $b \leqslant R$ (Fig. 38). We describe the circles (O, a) and (E, b) and denote their intersection points by X_1 and Y_1. If we now construct the segments $|OX| = 2^n|OX_1|$ and $|OY| = 2^n|OY_1|$, we obtain the required points of intersection X and Y of the given circles $(O, |AB|)$ and $(O_1, |CD|)$.

Proof.

$$|OX| = 2^n \cdot a = 2^n \cdot \frac{|AB|}{2^n} = |AB|,$$

$$|OY| = 2^n \cdot \frac{|AB|}{2^n} = |AB|.$$

From the similarity of the triangles OXO_1 and OX_1E ($|OX|/|OX_1| = |OO_1|/|OE| = 2^n$, the angle O_1OX is common) we have

$$|O_1X| = 2^n \cdot |EX_1| = 2^n \cdot \frac{|CD|}{2^n} = |CD|.$$

In exactly the same way we obtain $|O_1Y| = |CD|$.

Problem 28. Construct a point C_1 symmetric to a given point C with respect to a given straight line AB.

Construction. For the case $|AC| \leqslant R$ and $|BC| \leqslant R$ the construction is given in Problem 1. If the distance between the point C and the given straight line AB is less than R, then employing Problem 22 we can always find points A_1 and B_1 on the straight line, such that $|CA_1| \leqslant R$ and $|CB_1| \leqslant R$.

Now let the distance between the point C and the straight line AB be greater than R. We can take $|AB| < 2R$, otherwise we can find such points on the given straight line (Problem 22).

We take a point E in the plane such that $|CE| \leqslant R$ and the straight line CE intersects the segment AB. We construct the segment $|CD| = m|CE|$ ($|CE| = \ldots = |HD|$, Problem 2). We choose the point E and number m in such a way that the segments AD, AH, BD, and BH be less than R.

We construct the points D_1 and H_1 symmetric to the points D and H with respect to the given straight line (Problem 1). We construct the segment $|D_1C_1| = m|D_1H_1|$. The point C_1 is the required one, symmetric to the given

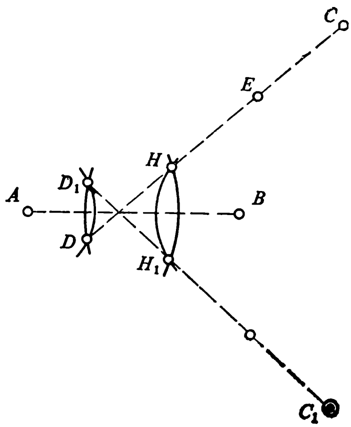

Fig. 39

point C with respect to the straight line AB (Fig. 39), i.e. $C_1 = S_{(AB)}(C)$.

Note. The solution to Problem 1 remains also valid in the general case if in order to construct the symmetrical point C_1 of intersection of the circles $(A, |AC|)$ and $(B, |BC|)$ we employ Problem 27.

Problem 29. (The fourth basic operation). Construct the points of intersection of a given circle $(O, |CD|)$ and a straight line given by two points A and B.

Construction. When the straight line does not pass through the centre of the circle we construct the point O_1 symmetric to the centre O of the given circle with respect to the straight line AB (Problem 28). We define the points of intersection X and Y of the circles $(O, |CD|)$ and $(O_1, |CD|)$ (Problem 27). The points X and Y are the required ones.

If the straight line passes through the centre of the circle* (Fig. 40) we construct the segment $r = |CD|/2^n$ on condition that $r \leqslant R/2$ (Problems 21 and 24). We describe the circle (O, r) and denote by A_1 and B_1 the points of its intersection with the circle (A, d) (or (B, d)), where d is an arbitrary radius less than or equal to R. If the circles (A, R) or (B, R) do not intersect the circle (O, r) when $d = R$ (in this case $|OA| > R + r$ and $|OB| > R + r$), then using Problem 22 we find a point E on the straight line AB such

* To check this fact see Note to Problem 1.

that $|OE| < R + r$; the circle (E, d) intersects (O, r) at the points A_1 and B_1. By varying the size of the radius d, we should make $a = |A_1B_1| \leqslant R/2$.

We bisect both arcs A_1B_1 of the circle (O, r) by the points X_1 and Y_1 (Problem 4). Then we construct the segments $|OX| = 2^n |OX_1|$ and $|OY| = 2^n |OY_1|$ (Problem 2,

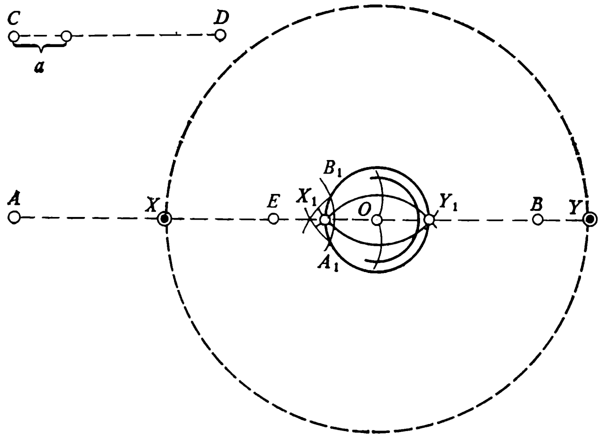

Fig. 40

$|OX_1| = |OY_1| = r \leqslant R/2$). The points X and Y are the required intersection points of the given straight line and the given circle.

The largest circle in this construction is drawn when the arc A_1B_1 is bisected. When an arc is bisected (see Problem 4), the radius of the largest circle equals $|BC| = \sqrt{2a^2 + r^2}$ (see Fig. 5). In our construction the radius satisfies the following inequality

$$\sqrt{2a^2+r^2} \leqslant \sqrt{2\left(\frac{R}{2}\right)^2 + \left(\frac{R}{2}\right)^2} < R.$$

Problem 30. There are three segments a, b, and c. Construct a segment x that would be the extreme term of the proportion $a/b = c/x$.

Construction. If $a \leqslant R$, $b \leqslant R$, and $c \leqslant R$, the construction is given in Problem 3.

Now let at least one of the above inequalities be invalid.

We construct segment $a_1 = a/2^n$, $b_1 = b/2^n$, and $c_1 = c/2^m$ (Problems 21 and 24), selecting natural numbers n and m such that $a_1 \leqslant R$, $b_1 \leqslant R$, $c_1 \leqslant R$, and $c_1 \leqslant 2a_1$.

We construct the segment x_1 that would be the extreme term of the proportion $a_1/b_1 = c_1/x_1$. If we now construct the segment $x = 2^m \cdot x_1$ (Problems 2 and 25), we find the required segment, which is the extreme term of the proportion $a/b = c/x$.

Proof. The proportion

$$\frac{a}{2^n} : \frac{b}{2^n} = \frac{c}{2^m} : x_1.$$

can be written as

$$a/b = c/2^m x_1.$$

Problem 31. (The fifth basic operation). Construct the intersection point of the given straight lines AB and CD, each of which is defined by two points.

Construction of the intersection point of the given straight lines by compasses with a bounded opening is just the same as in Problem 7. However, instead of Problems 1 and 3, we make use of Problems 28 and 30, respectively. To find point E and the required point X (the intersection point of the straight lines AB and CD) we employ Problem 27.

Note. Making use of Problem 22 we can take the points A, B, C, and D, which define the given straight lines, so close to each other that all the circles drawn in this construction will have radii not greater than R and so can be drawn by compasses with a bounded opening of the legs.

* * *

On the basis of the above discussion we come to the following conclusion.

All five basic operations (the simplest problems) can be carried out (solved) with compasses describing circles whose radii do not exceed some prescribed segment R.

Each geometrical construction problem solvable by compasses and a ruler can always be reduced to a finite sequence of basic operations in a certain order (Sec. 1).

Thus, the following theorem holds.

Theorem. *All geometrical construction problems solvable by compasses and a ruler can be solved exactly using only com-*

passes and describing circles whose radii do not exceed a certain prescribed segment.

We now investigate a general method of solving construction problems by compasses the opening of whose legs is bounded from above by the segment R.

Suppose that it is required to solve a certain construction problem, solvable by compasses and a ruler, using only compasses with a bounded opening. Let us imagine this problem solved by compasses alone in the classical sense when the opening is bounded in no way. As a result, we obtain a certain figure Φ which consists only of a finite number of circles. We denote by R_1 the largest of the radii of all the circles constituting the figure Φ. If it turns out that $R_1 \leqslant R$, then the indicated construction can be carried out by means of the compasses with a bounded opening.

Now let $R_1 > R$. Let us take a natural number n such that $R_1/2^n \leqslant R$. Now if we reduce all the segments prescribed, including those which define the radii of the given circles, by a factor of 2^n and solve the problem by the given compasses, we obtain the figure Φ'. It will be similar to the figure Φ, the ratio of similitude (homothetic ratio) being $1/2^n$. All the circles of the figure Φ' can be described with the given compasses since their radii are no greater than $R_1/2^n$ ($R_1/2^n \leqslant R$). It should be noted here that if, according to the conditions of a problem, a figure Φ_1 is given in the plane of the drawing, then it is necessary to take one of the points of the figure as the centre of similitude and to construct a similar figure Φ_1' with the ratio of similitude $1/2^n$ (that is, to make the figure Φ_1 2^n times smaller)*.

Let us denote by Ψ' that part of the figure Φ which is taken as the required result. We construct the figure Ψ similar to the figure Ψ' with the centre of similitude O and the ratio of similitude 2^n (we enlarge the figure Ψ' 2^n times). To this end we construct the segments $[OX_1]$, . . ., $[OX_k]$ such that

$$|OX_1| = 2^n\,|OX_1'|,\quad |OX_2| = 2^n|OX_2'|,\quad \ldots,\quad |OX_k| = 2^n|OX_k'|,$$

where X_1', X_2', . . ., X_k' denote all the intersection points of the circles in the figure Ψ'. The points X_1, X_2, . . ., X_k of the figure Ψ denote the centres and intersection points of the circles which make up the figure.

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* The data in a problem can be given by several figures Φ_1, Φ_2, . . .

The figure Ψ represents the required result of the given problem. Straight lines and circles whose radii are greater than R cannot be drawn in the figure Ψ with the given compasses. They can be constructed in the form of points, arbitrarily close to each other (Problems 22 and 26).

To illustrate the above arguments we use the solution of Problem 27 where Φ consists of the circles $(O, |AB|)$ and $(O_1, |CD|)$. We are given three segments AB, CD, and OO_1, where $|AB|$ and $|CD|$ are the radii of two circles and $|OO_1|$ is the distance between their centres O and O_1. These circles are, respectively, the figures Φ_1, Φ_2, and Φ_3 given in the statement of the problem (see Fig. 38). The figures Φ_1', Φ_2', and Φ_3' are the segments a, b, and OE, respectively, with the centres of similitude A, C, and O and the ratio of similitude $1/2^n$. The figure Φ' incorporates the circles (O, a) and (E, b) (together with their centres O and E). The figure Ψ', taken as the required result in the figure Φ', consists of two points X_1 and Y_1. The result of the solution is the figure Ψ consisting of points X and Y. The point O is the centre of similitude (in Fig. 38 $2^n = 4$, $n = 2$).

When solving construction problems the number n is usually unknown, since we cannot construct the figure Φ with the given compasses, which means that we cannot know the radius R_1 of the largest of the circles. Taking this circumstance into account, we carry out the solution of the problem with the given compasses (with a bounded opening) until we come to a circle with the radius $r_1 > R$. We select the natural number n_1 in such a way that $r_1/2^{n_1} \leqslant R$. We diminish the given segments 2^{n_1} times and begin the solution of the given problem once more. As a result we either solve the problem completely and construct the figure Φ', or again arrive at a circle of radius $r_2 > R$. We select the natural number n_2 in such a way that $r_2/2^{n_2} \leqslant R$ and again diminish the segments 2^{n_2} times (here the segments given in the statement of the problem are reduced $2^{n_1+n_2}$ times). Then we begin to solve the problem for the third time and so on. After a finite number k of steps the figure has been constructed (the original segments given in the statement of the problem have been reduced $2^{n_1+n_2+\ldots+n_k}$ times).

Using a general method of solution, it is easy to construct by compasses with a bounded opening the figures inverse of a given point, a straight line, or a circle (Problems 15-18).

In conclusion to this section we give the solution of the following problem.

Problem 32. Divide the given segment AB into five equal parts if we cannot have a segment five times as large as the given segment $|AB| = a$.

In the extensive work of Mascheroni [9] this problem is the only one solved with the constraint indicated in its condition.

Construction. We describe the circle (B, a) and construct the point E diametrically opposite to the point A (in

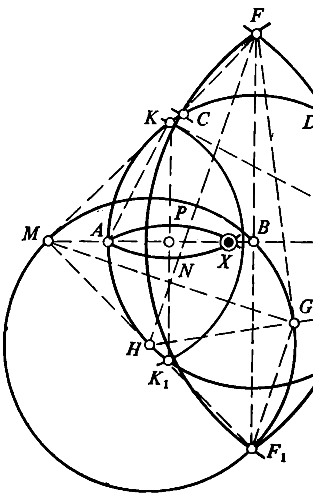

Fig. 41

Fig. 41 $|AC| = |CD| = |DE| = a$). We draw the circles $(A, |AD|)$ and $(E, |EC|)$ until they meet at the points F and F_1. We mark the point H of intersection of the circles (F_1, a) and (B, a). We then describe the circles $(H, |HF_1|)$ and $(F, |AE|)$ and at their intersection we obtain the point G. The circle $(A, |F_1G|)$ intersects the circle (B, a) at the points K and K_1. If now the circles $(K, |AK|)$ and $(K_1, |AK_1|)$ are drawn, then the required point X is obtained, i.e.

$$|BX| = \frac{1}{5}|AB|.$$

Proof. Let the point M on the circle $(H, |HF_1|)$ be diametrically opposite to the point F_1 and let the point N be the intersection point of the straight lines HF and MG. The segment $|BF| = |BF_1| = \sqrt{2}a$. The length of the tangent emerging from the point F to the circle $(H, |HF_1|)$ is equal to

$$b = \sqrt{|FF_1| \cdot |FB|} = \sqrt{2\sqrt{2}a\sqrt{2}a} = 2a.$$

But, on the other hand, $|FG| = 2a$ by construction, hence, the straight line FG touches the circle $(H, |HF_1|)$ at the point G.

From the right-angled triangle FGH we have

$$|HF| = \sqrt{|HG|^2 + |GF|^2} = \sqrt{5}a.$$

The triangle FMF_1 is isosceles, since the angle F_1BM is a right angle, inscribed and subtended by the diameter F_1M of the circle $(H, |HF_1|)$. This means that $|MF_1| = |MF| = 2a$.

The triangle MGF is also isosceles ($|MF| = |FG| = 2a$), therefore (MG) is perpendicular to (HF).

From the right-angled triangle HGF where GN is the height, we obtain

$$|HG|^2 = a^2 = |HF| \cdot |HN| = \sqrt{5}a\,|HN|$$

or

$$|HN| = \frac{a}{\sqrt{5}}.$$

From the right-angled triangles HNG and MGF_1 we find

$$|NG| = \sqrt{a^2 - \left(\frac{a}{\sqrt{5}}\right)^2} = \frac{2a}{\sqrt{5}} = \frac{1}{2}|MG|,$$

$$|GF_1|^2 = 4a^2 - \frac{16}{5}a^2 = \frac{4}{5}a^2.$$

And, finally, from the right-angled triangle AKE we have

$$|AK|^2 = |GF_1|^2 = |AE| \cdot |AP|$$
$$= 2a \cdot \frac{|AX|}{2} = |AX| \cdot a,$$

or

$$|AX| = \frac{|GF_1|^2}{a} = \frac{4a}{5}.$$

Hence

$$|BX| = \frac{1}{5}|AB|.$$

Sec. 6. Constructions by Compasses Only when the Opening of the Legs Is Bounded from Below

In this section we use compasses the opening of whose legs is bounded *only from below* by the prescribed segment $R_{\min}$. With such compasses it is possible to draw circles of any radius greater than or equal to the segment $R_{\min}$. In the following we will simply write R instead of $R_{\min}$.

Problem 33. Construct a segment n times greater than a given segment AA_1.

Construction. We construct the segment A_1E perpendicular to the given segment AA_1 (Problem 8, we take $|OA| \geqslant R$). We define the point E', symmetric to the point E with respect to the straight line AA_1 (Problem 1, here $|AE| > R$ and $|A_1E| > R$). We construct the point A_2, symmetric to the point A with respect to the straight line EE'. The segment $|AA_2| = 2|AA_1|$ (Fig. 42).

Then we construct $[A_2E_1] \perp [AA_2]$ and $E_1' = S_{(AA_2)}(E_1)$. If we now construct the points $A_3 = S_{(E_1E_1')}(A_1)$ and $A_4 =$

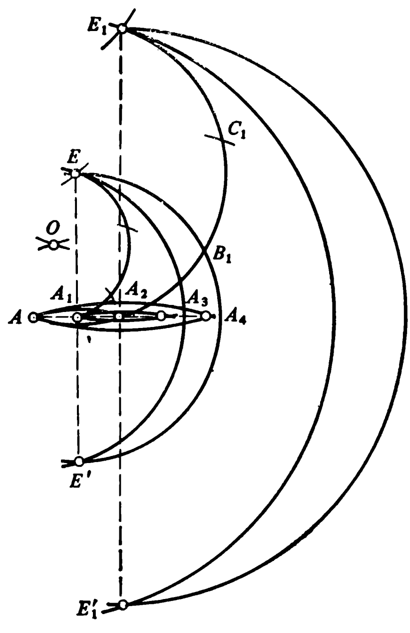

Fig. 42

$S_{(E_1E_1')}$ (A), we have

$$|AA_3| = 3\,|AA_1|,\quad |AA_4| = 4\,|AA_1|.$$

Further the construction is performed in a similar way once again.

For $|AA_1| \geqslant R$ the construction is given in Problem 2. The radii of all the circles drawn in this construction are not less than the segment R.

Note. From the given construction it is obvious that the points A_2, A_4, A_8, A_{16}, . . . can be constructed immediately missing out the construction of the points A_3, A_5, A_6, A_7, A_9, i.e. the segments 2, 4, 8, 16, . . ., 2^n times greater than the given segment AA_1 can be found.

Problem 34. Construct a segment n times smaller than the given segment AB (divide a segment into n equal parts).

Construction. For $|AB| \geqslant R$ the construction is given in Problem 9.

Now let $|AB| < R$. We construct the segment $|AB'| = m\,|AB|$ (Problem 33) and take a natural number m such that $|AB'| \geqslant R$. We divide the segment into nm equal parts (Problem 9). We obtain the required segment AX:

$$|AX| = \frac{|AB'|}{nm}.$$

Indeed

$$|AX| = \frac{|AB'|}{nm} = \frac{m\,|AB|}{nm} = \frac{|AB|}{n}.$$

Note. In this case if we use the construction of Problem 10 instead of that of Problem 9, we get

$$|AX| = \frac{1}{2^n}\,|AB|.$$

The solution of Problem 5 remains valid for compasses with opening bounded from below.

Problem 35. (The second basic operation). Describe a circle of radius $r = |AB|$ with the given point O as centre.

Construction. If $|AB| \geqslant R$, then the circle can be drawn directly with the given compasses. But if $|AB| < R$, then we cannot describe the circle as a continuous curve with the given compasses; in this case it is possible to construct any number of points situated as closely together as desired on the circumference of the circle defined by its centre and radius.

Let $|AB| < R$. We describe the circles (O, a) and (A, a) of an arbitrary radius $a > R + r$ and take two points C and D on the second circle, such that $|CD| \geqslant R$. We take the point C_1 on (O, a) and describe the circle $(C_1, |CD|)$ which intersects the circle (O, a) at the point D_1. If now we describe the circles $(C_1, |CB|)$ and $(D_1, |BD|)$, we obtain at their intersection the point X which lies on the required

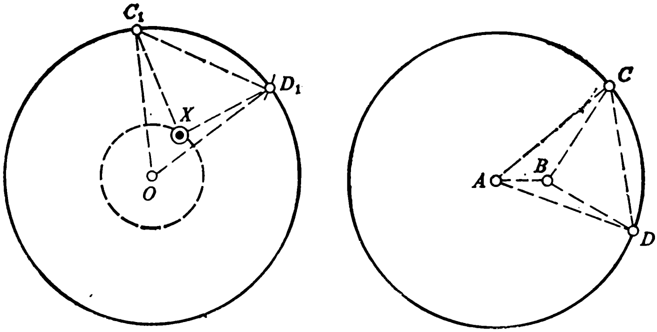

Fig. 43

circle (O, r). By varying the position of the chord C_1D_1 on the circle (O, a), it is possible to construct any number of points of the given circle (Fig. 43).

The validity of the construction immediately follows from the congruence of the triangles ACD, OC_1D_1, and BCD, XC_1D_1.

Now we consider a general method of solving geometrical construction problems using only compasses whose opening is bounded below by a segment R. By this method it is possible to solve every construction problem solvable by compasses and a ruler, including the third, fourth, and fifth basic simplest problems (Problems 5-7).

The general method of solving problems by compasses alone which describe circles of radius not less than R coincides with the general method described in Sec. 5. The difference between these methods is that the segments given in the conditions of the problem have not to be diminished 2^n times but, quite the reverse, increased n (or 2^n) times (Problem 33).

Let us assume that a certain construction problem, solvable by compasses and a ruler, should be solved by compasses alone whose opening is bounded from below by a segment R.

Let us imagine this problem being solved by compasses alone whose opening is not bounded. As a result we obtain a figure Φ which consists only of circles. We denote by R_1 the smallest of the radii of all the circles comprising the figure Φ. We select a natural number n such that $nR_1 \geqslant R$ and construct the figure Φ' similar to the figure Φ and n times larger. We denote by Ψ' the part of the figure Φ' which represents the required result. If now we construct the figure Ψ which is n times smaller than the figure Ψ' we find the required result of our problem (Problem 34).

Thus, we arrive at the following theorem.

Theorem. *All geometrical construction problems solvable by compasses and a ruler can be solved exactly using only compasses capable of describing circles whose radii are not less than a prescribed value.*

Sec. 7. Constructions by Compasses Only When the Opening Is Fixed

Geometrical constructions by compasses with a fixed opening, capable of describing only circles of constant radius R, were investigated by many scholars. A large part of the work *The Book of Geometrical Constructions* by the Arab mathematician Abu Yaf is devoted to this subject. Leonardo da Vinci, Cardano, Tartaglia, Ferrari, and others have been engaged in solving construction problems using only compasses with a fixed opening.

By compasses with a fixed opening R we can construct a straight line perpendicular to the segment AB at one of its ends, only if $|AB| < 2R$ (Problem 8); we can increase the segment R 2, 3, 4, . . . times (Problem 2). If $|AB| < 2R$ and $|AB| \neq R$ it is possible to construct the points of a straight line AB (Problem 5) changing the position of the symmetric points C and C_1 each time. However, with these compasses we cannot divide segments and arcs into equal parts, find proportional segments, and so on.

Thus, *it is impossible to solve all construction problems, solvable by compasses and a ruler, using only compasses with a fixed opening.*

In two preceding sections we considered the solutions of construction problems using only compasses when certain constraints were imposed on the opening, and suggested general methods of solving the problems by such instruments,

Naturally, the question arises: Is it possible to solve construction problems by compasses whose opening is bounded from above and from below simultaneously, i.e. by compasses capable of describing circles of radius not smaller than R_{min} and not greater than R_{max}.

The answer to this question is in the work of Japanese mathematician K. Yanagihara [14]. K. Yanagihara proved that all construction problems solvable by compasses and a ruler can be solved exactly using only compasses also in the case when a radius is simultaneously bounded from above and from below by the lengths R_{max} and R_{min}.

This proof is too complex and abstract to be given in our book.

The difference $R_{max} - R_{min}$ in Yanagihara's basic theorem can be taken sufficiently small. In other words, all geometrical construction problems solvable by compasses and a ruler can be solved exactly using only compasses with a "nearly" constant opening. And, as has already been noted at the beginning of this section, all these problems cannot be solved by compasses with a fixed opening.

Sec. 8. Constructions by Compasses Only on Condition that All Circles Pass Through the Same Point

In this section we consider the solution of geometrical construction problems by compasses alone on condition that all the circles being drawn pass through the same point in the plane*.

Definition. *The angle of intersection of two circles* (in the general case of two curves) is understood to be the angle made by the tangents to the circles (curves) at the point of intersection. *The circles* are said to be *orthogonal* if they intersect at right angles.

Theorem 1. *If the circle* (O_1, R) *and the circle of inversion* (O, r) *are orthogonal, then the former is its own inverse***.

Proof. If the circles are orthogonal, then the angle OAO_1 formed by their radii at the intersection of the circles is a right angle. This means that the straight line OA is a tan-

---

* In this section no constraints are imposed on the openings of the compasses.

** The converse theorem is also true but it is not used in this section.

gent to the circle (O_1, R) at the point A, and

$$|OP|\cdot|OP'| = |OA|^2 = r^2.$$

The last equation is true for any secant OP. The point P' is the inverse of the point P. The arc APA_1 of the circle (O_1, R) is the inverse of the arc $AP'A_1$ (Fig. 44).

In Problem 11 we gave the construction of a segment 3^n times as great as the given segment AA_0. In this construction all the circles pass through the point A. The only exception

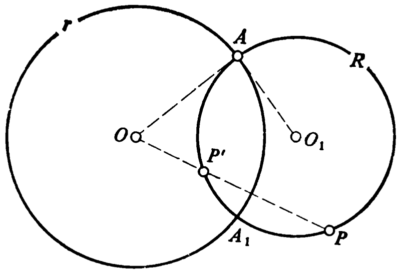

Fig. 44

is the circle $(A, |AA_0|)$ which is drawn in order to find the points E and E' and does not pass through the point A (Fig. 18).

However, we may not draw the circle $(A, |AA_0|)$ but proceed as follows.

We make the opening of the compasses equal to $|AA_0|$ and set the point of the pencil at the point A (actually this point of the pencil differs in no way from the sharp end of the needle of the other leg of the compasses). Then, without changing the opening of the compasses we set the second leg in such a way that the needle point falls on the arc of the circle $(A_0, |AA_0|)$. The needle point of the compasses gets into the point E. If we now describe the circle $(E, |AE|)$, then at its intersection with the circle $(A_0, |AA_0|)$ we obtain the point C. Just in the same way it is possible to construct the point C'. We introduce this *additional operation* into the theory of geometrical constructions by compasses alone.

And so it is possible to construct a segment 3^n times as great as the given segment (Problem 11) in such a way that all the circles pass through the same point,

The solutions of Problems 15, 16, and 17 are found in such a way that all the circles pass through the same point O which is the centre of inversion (see Figs. 26, 28, and 29).

Consider the construction of the point X, which is the inverse of the point C when $|OC| \leqslant r/2$ (Problem 15). In order for all the circles without exception to pass through one point O, it is necessary to construct the segment $|OC_1| = 3^n |OC| > r/2$ (Problem 11, the remarks made at the beginning of this section being taken into account) instead of the segment $|OC_1| = n|OC| > r/2$ (Fig. 27) and to construct $|OX| = 3^n |OC_1'|$.

Thus, with the help of compasses alone, it is possible to construct: the inverse of a given point; a circle passing through the centre of inversion which is the inverse of a given straight line; and, finally, a straight line (two of its points) which is the inverse of the circle that passes through the centre of inversion O. In each construction we draw only the circles passing through the same point O, the centre of inversion.

* * *

As we noted in Introduction, J. Steiner showed that all construction problems solvable by compasses and a ruler, can also be solved exactly by a ruler alone if in the plane of the drawing there is a constant (auxiliary) circle (O_1, R) and its centre.

Now, let us suppose that a certain construction problem was solved by Steiner's method; as a result we obtain a figure Φ in the plane of the drawing which consists, apart from the auxiliary circle, of straight lines only. Let us take an arbitrary circle (O, r), the only condition being that its centre O lie neither on the circle (O_1, R) nor on straight lines of the figure Φ and let us take it as the circle of inversion. We construct the figure Φ' which is the inverse of the figure Φ. The figure Φ' so constructed consists of circles only, all of which (with the exception of the circle of inversion (O, r) and the circle which is the inverse of Steiner's circle (O_1, R)) pass through the same point O, i.e. the centre of inversion.

If the circle of inversion (O, r) intersects the auxiliary circle (O_1, R) at right angles, then by virtue of Theorem 1, Steiner's circle (O_1, R) is self-inverse (i.e. it simultaneously belongs to Φ and Φ'). The figure Φ consists of straight lines and Steiner's circle; the inverse figure Φ' includes Steiner's

circle, circles passing through the centre of inversion, and, perhaps, some isolated points (centres of these circles). To construct Φ' it is necessary to make use of Problems 15 and 16 only.

Thus, in the construction of the figure Φ', the inverse of the figure Φ (when the circle of inversion and Steiner's circle are orthogonal), all the circles used, pass through the same point O, there being only two exceptions, namely, the circle of inversion (O, r) and the auxiliary Steiner's circle (O_1, R).

We should like to note that the order of operation in the construction of the figure Φ' should be the same as in the construction of the figure Φ by Steiner's method.

In order to illustrate the above, let us solve the following problem.

Problem 36. Given the straight line AO_1 and the point C outside it. Construct a straight line passing through C and perpendicular to the straight line AO_1, and find the point of their intersection.

Construction. Let (O_1, R) be Steiner's circle where $R = |O_1A|$. We construct the figure Φ by Steiner's method in three steps. (1) We draw the straight line AO_1 until it intersects the circle (O_1, R) at point B. (2) We draw the straight lines AC and BC and mark the points E and D of their intersection with Steiner's circle. (3) If we now draw the straight lines AD and BE, we obtain the point F at their intersection. The straight line CF is perpendicular to the straight line AO_1. Let H be the intersection point of the straight lines AB and CF (Fig. 45).

Proof. The segments CD and EF are the heights of the triangle ACF, since the angles ADB and AEB are right angles (they are inscribed into the circle (O_1, R) and are subtended by the diameter AB), therefore (FC) is perpendicular to (AO_1) because the heights of the triangle intersect at the same point.

The figure Φ in this problem consists of Steiner's circle (O_1, R) and six straight lines AO_1, AC, AD, CD, CF, and EF. At the beginning of the construction the figure consists of Steiner's circle, the straight line AO_1, and the point C.

Now let us consider the solution of Problem 36 by compasses alone, on condition that all the circles drawn, with the exception of two, will pass through the same point.

Construction. First we define the circle (O, r) so that it is

orthogonal to Steiner's circle (O_1, R), where $R = |O_1A|$. To do this we take two arbitrary points K and M on the circle (O_1, R) and construct (KO) perpendicular to $[KO_1]$ (Problem 8, 2nd method) for which we draw the circles $(M, |KM|)$ and $(K, |KP|)$ until they meet at the point O. The point P is obtained at the intersection of the circles $(M, |KM|)$ and (O_1, R) (see Fig. 45).

Varying the position of the points K and M it is possible to find a point O which does not lie on any of the straight

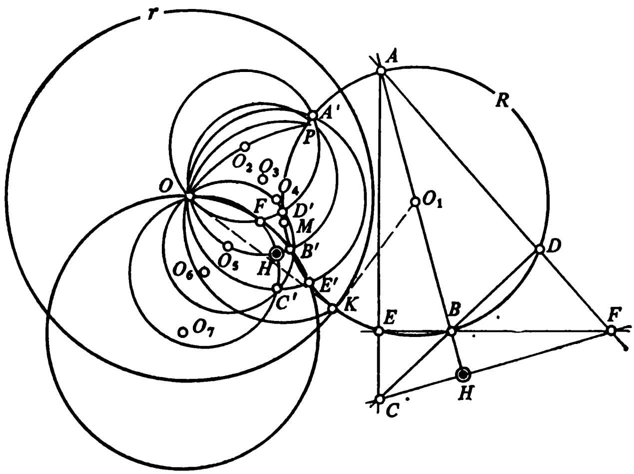

Fig. 45

lines of the figure Φ. The circle $(O, |OK|)$ is orthogonal to Steiner's circle (O_1, R). Therefore, if the circle (O, r), where $r = |OK|$ is assumed to be the circle of inversion, then Steiner's circle (O_1, R) is inverse to itself.

The figure Φ', inverse of the figure Φ, is built up from seven circles $(O_2, |OO_2|)$, $(O_3, |OO_3|)$, $(O_4, |OO_4|)$, $(O_5, |OO_5|)$, $(O_6, |OO_6|)$, $(O_7, |OO_7|)$, and (O_1, R)*. (Steiner's circle is self-inverse, therefore it belongs to the figures Φ and Φ' simultaneously.) The first six circles pass through the centre of inversion O and they are respectively inverse to

* The centres of these circles O_2, O_3, O_4, O_5, O_6, O_7, and O_1 can be regarded as separate isolated points of the figure Φ'.

the straight lines AF, AO_1, AC, CD, CF, and EF of the figure Φ. The points A', B', C', D', E', F' and H' of the figure Φ' are respectively inverse to the points A, B, C, D, E, F, and H of the figure Φ. All the circles drawn to construct the first six circles of the figure Φ', and also the circles $(M, |KM|)$ and $(K, |KP|)$ needed to define the centre of inversion O, pass through the same point O in the plane.

At the beginning of the construction the figure Φ consists of the given point C and the straight line AO_1, we add Steiner's circle (O_1, R) to it. In order to construct the other straight lines of the figure Φ by compasses alone, i.e. to construct the points B, E, D, and the required points F and H, we construct the figure Φ' reproducing, in fact, three steps of Steiner's construction.

(1) We construct the circle $(O_3, |OO_3|)$ inverse to the straight line AO_1 (Problem 16) which intersects Steiner's circle (O_1, R) at the points A' and B' (the point A' is inverse to the point A). We find the point B, inverse to the point B' (Problem 15). The point B is the intersection point of the straight line AO_1 and the circle (O_1, R) (see Fig. 45).

(2) We construct the circles $(O_4, |OO_4|)$ and $(O_5, |OO_5|)$ which are inverse to the straight lines AC and BC. Let these circles intersect Steiner's circle at the points A' and E', B' and D', respectively. We construct the points E and D, inverse to the points E' and D'.

(3) Finally, we construct the circles $(O_2, |OO_2|)$ and $(O_7, |OO_7|)$ which are inverse to the straight lines AD and BE and intersect at the point F'. We find the point F inverse to the point F'. CF is the required straight line. Now let the circle $(O_6, |OO_6|)$, inverse to the straight line CF, intersect the circle $(O_3, |OO_3|)$ at the point H'. The point H, inverse to the point H', is the required point, i.e.

$$H = (CF) \cap (AO_1).$$

On the basis of the above we can formulate the theorem.

Theorem 2. *Every geometrical construction problem solvable by compasses and a ruler can be solved by compasses alone in such a way that all circles of the construction except two (the circle of inversion and the auxiliary Steiner's circle) pass through the same point, that is, the centre of inversion.*

* * *

Now, let a certain problem be solved by Steiner's method. As a result we obtain the figure Φ which consists of the circle (O_1, R) and straight lines, some of which pass through the centre O_1 of the auxiliary circle. If Steiner's circle (O_1, R) is taken as the circle of inversion and the figure Φ', inverse of Φ is constructed, then the figure Φ' is built up from *straight lines* and *circles*, and, moreover, all these straight lines and circles, with the exception of the circle (O_1, R), pass through the same predetermined point O_1*.

Hence follows the theorem.

Theorem 3. *Every geometrical construction problem can always be solved by a ruler and compasses in such a way that all straight lines and circles except one (the circle of inversion, which is also the auxiliary Steiner's circle here) pass through one predetermined point which is the centre of inversion.*

* * *

Suppose now that when solving geometrical construction problems by compasses alone we can use a ruler only once (in other words there is the straight line AB drawn by a ruler in the plane of the drawing). Let us take an arbitrary circle (O, r) with centre O, not on the straight line AB, as the circle of inversion, and let us construct the circle (O_1, R) inverse to the given straight line (Problem 16). The circle (O_1, R) passes through the centre of inversion O while $R = |OO_1|$.

The solution of any construction problem by Steiner's method with the use of the auxiliary circle (O_1, R) gives the figure Φ which consists only of straight lines and the circle (O_1, R). The inverse figure Φ' consists, apart from the straight line AB, of circles only, passing through the centre of inversion O. At the same time we assume that none of the

* A. Adler ([1], Sec. 20) states that if the auxiliary Steiner's circle (O_1, R) is taken as the circle of inversion, then "Not only is it possible, as had been shown by Mascheroni, to solve all geometrical construction problems of the second degree exclusively by compasses, but even to add the condition that all the circles included in the construction, except one of them, should pass through the same arbitrarily selected point".

The inaccuracy of this statement follows from the fact that all construction problems solvable by compasses and a ruler cannot be solved by a ruler alone, if the centre of the auxiliary circle O_1 is unknown, i.e. if no straight lines are passed through the centre O_1 (the lines being self-inverse (see Theorem 2, Sec. 3) and, therefore, belonging to the figure Φ').

straight lines of the solution by Steiner's method had passed through the point O, lying on the auxiliary circle (O_1, R), otherwise another circle should be taken as the circle of inversion (O, r).

If the straight line AB has not been drawn, but a single use of a ruler is permitted, then we take an arbitrary circle (O_1, R) in the plane of the drawing as an auxiliary and we solve the given problem by Steiner's method. Then we take a point O on the circumference of that circle on condition that it should not lie on any of the straight lines of the figure Φ. We describe the circle (O, r) of the radius $r < 2R$ and denote its points of intersection with the circle (O_1, R) by A and B. We take the ruler and draw the straight line AB which is the inverse of the circle (O_1, R) if we regard (O, r) as the circle of inversion. Then we construct the figure Φ', which is the inverse of the constructed figure Φ.

Theorem 4. *If a straight line is drawn in the plane of the drawing, then all construction problems solvable by compasses and a ruler can be solved by compasses alone in such a way that all circles of this constructions, except one (the circle of inversion) pass through the same point of the plane.*

This theorem is to a certain extent analogous to Steiner's basic theorem for constructions by a ruler alone given a fixed circle.

* * *

Now let there be drawn by a ruler a certain figure Ψ which consists of straight lines and segments (for instance, two carallel lines or a parallelogram, and so on).

Let us suppose that we solved a certain construction problem employing Steiner's method, taking Ψ as an auxiliary figure. As a result we obtain a certain figure Φ which is built up from straight lines only. The figure Ψ is a subset of the figure Φ.

Let us take an arbitrary circle (O, r) on condition that its centre does not lie on any of the straight lines of the figure Φ, as the circle of inversion, and let us construct the figure Φ', inverse of the figure Φ. The figure Φ' consists only of the circles passing through the same point O, the centre of inversion.

Theorem 5. *If a certain figure in the plane is given (drawn), consisting only of straight lines and segments, then all construc-*

tion problems which can be solved by Steiner's method with this figure as an auxiliary one, can always be solved by compasses alone in such a way that all circles except one (the circle of inversion) pass through the same point, taken at random in the plane of the drawing.

* * *

At the beginning of this section we introduced the *additional operation* by means of which Problem 15 can always be solved in such a way that all the circles in the construction except one (the circle of inversion (O, r)) pass through one predetermined point O, the centre of inversion.

But if this additional operation is not applied, then, to construct the points E and E', E_1, and E_1', E_2 and E_2', and so on in Problem 11 which is used in Problem 15 it is required to construct the circles $(A, |AA_i|)$, where $i = 1, 2, \ldots$ In this case in Theorem 2 the phrase, 'except two (the circle of inversion and the auxiliary Steiner's circle)' should be replaced by: 'except the auxiliary Steiner's circle (O_1, R), the circle of inversion (O, r), and, perhaps, a few more concentric circles (O, r_i), $i = 1, 2, \ldots$'.

In Theorems 3, 4, and 5 the phrase 'except one (the circle of inversion)' should be replaced by 'except the circle of inversion (O, r) and, perhaps, a few more concentric circles (O, r_i), $i = 1, 2, \ldots$.

For instance, Theorem 3 can be formulated as follows.

Theorem 3′. *Every geometrical construction problem can always be solved by a ruler and compasses in such a way that all straight lines and circles pass through one predetermined point O except the circle of inversion (O, r) and, perhaps, a few more concentric circles (O, r_i), $i = 1, 2, \ldots$, with the centre of inversion O as a centre.*

Thus, all the circles of the construction can be divided into two groups: the circles of the first group pass through the same point O, and the circles (one or more) of the second group are described from the same point O as a centre. Among these concentric circles there is always the circle of inversion (O, r).

APPENDICES

Appendix 1. Symbols and Notation Used in the Book

N	The set of natural numbers
Φ, Ψ	Geometrical figures
(AB)	A straight line passing through the points A and B
$[AB]$	A segment with the end points A and B
$[AB)$	A ray AB
$\parallel$	Parallel
$(AB) \parallel (CD)$	A straight line AB is parallel to a straight line CD
$\perp$	Perpendicular
$[AB] \perp [CD]$	A segment AB is perpendicular to a segment CD
$\angle$	Angle, $\angle ABC$ is an ABC angle
$\widehat{\ }$	Magnitude of an angle, $\widehat{ABC} = 60°$
$\smile$	Arc, $\smile AB$ is an AB arc
$\smile$	Angular value of an arc, $\overset{\smile}{CD} = 90°$
$\triangle$	Triangle
$\sim$	Sign of similarity, $\triangle ABC \sim \triangle CDE$
$\cong$	Sign of congruence, $\triangle ABC \cong \triangle CDE$
$\in$	Sign of membership, $A \in (CD)$ means that the point A belongs to (CD)
$\subset$	Sign of inclusion, $[AB] \subset \Phi$ means that a segment AB belongs to a figure Φ
$\cup$	Sign of union
$\cap$	Sign of intersection, $E = (AB) \cap (CD)$ means that straight lines intersect at the point E
$\varnothing$	Empty set, $M = \varnothing$ means that the set M is empty
$A\ (x)$	A point with the coordinate x
$M = \{a;\ b;\ c;\}$	The set M consists of the elements a, b, and c
$(O,\ r)$	A circle or a circumference with the centre O and the radius r
$\overrightarrow{AB}$	Vector AB

$\uparrow\uparrow$	Similarly directed vectors, $\overrightarrow{AB}\uparrow\uparrow\overrightarrow{CD}$ means that the vectors are on parallel straight lines and similarly directed
$\uparrow\downarrow$	Oppositely directed vectors
$S_{(AB)}\,(C)$	A point symmetric to the point C with respect to the straight line AB
$\mathbf{H}_O^k$	Homothety (similarity) with the centre O and coefficient k

Appendix 2. Proof for Problem 18 in the General Case

Problem 18 includes a proof for the case when the centre of inversion O lies outside the given circle (O_1, R). The proof

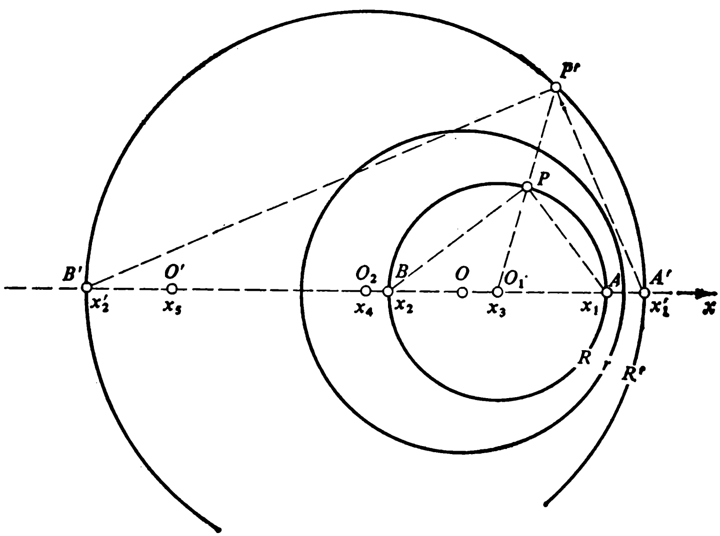

Fig. 46

does not require the construction of the tangent OP and is valid in the general case.

Proof. Let us introduce the coordinate line: O (the centre of inversion) is the origin, the abscissa passes through the centre of the given circle, O_1. The coordinates of the points on the x-axis are denoted by $A\ (x_1)$, $A'\ (x_1')$, $B\,(x_2)$, $B'\ (x_2')$, $O_1\ (x_3)$, $O_2\ (x_4)$, and $O'\ (x_5)$ (see Figs. 25, 30, and 46).

Then

$$R=\frac{|x_2-x_1|}{2}, \quad R'=\frac{|x_2'-x_1'|}{2}, \tag{1}$$

$$|x_3|=\frac{|x_1+x_2|}{2}, \quad |x_4|=\frac{|x_1'+x_2'|}{2}, \tag{2}$$

$$|x_1|\cdot|x_1'|=|x_2|\cdot|x_2'|=r^2. \tag{3}$$

Let the point O' be inverse to the point O_2 (the centre of the required circle which is inverse to the given circle), i.e.

$$|x_4|\cdot|x_5|=r^2. \tag{4}$$

It is necessary to prove that

$$|x_3|\cdot|x_5-x_3|=R^2, \tag{5}$$

i.e. that the point O' is inverse to the point O if the circle (O_1, R) is taken as the circle of inversion.

Indeed, from Eqs. (1)-(4) we have

$$\begin{aligned}
|x_3|\cdot|x_5-x_3| &= \frac{1}{2}|x_1+x_2|\cdot\left|\frac{1}{2}|x_1+x_2|-\frac{r^2}{|x_5|}\right| \\
&= \frac{1}{2}|x_1+x_2|\cdot\left|\frac{|x_1+x_2|}{2}-\frac{2r^2}{|x_1'+x_2'|}\right| \\
&= \frac{|x_1+x_2|}{2}\cdot\left|\frac{|x_1+x_2|}{2}-\frac{2r^2}{\left|\frac{r^2}{x_1}+\frac{r^2}{x_2}\right|}\right| \\
&= \frac{|x_1+x_2|}{2}\cdot\left|\frac{|x_1+x_2|^2-4|x_1|\cdot|x_2|}{2|x_1+x_2|}\right| \\
&= \frac{1}{4}\Big||x_1+x_2|^2-4|x_1|\cdot|x_2|\Big|=\frac{1}{4}|x_2-x_1|^2=R^2.
\end{aligned}$$

The theorem has been proved.

REFERENCES

1. A. Adler, *Teorie der Geometrischen Construktionen*, Leipzig, 1906.
2. I. I. Aleksandrov, *Collection of Problems on Geometrical Constructions*, Uchpedgiz, Moscow, 1950 (in Russian).
3. B. I. Argunov and M. B. Blank, *Geometrical Constructions on a Plane*, Uchpedgiz, Moscow, 1957 (in Russian).
4. E. Ya. Astryab and S. O. Smogorzhevskii (editors), *Method for the Solution of Geometrical Construction Problems in Secondary Schools*, Radyan'ska shkola, Kiev, 1940 (in Ukrainian).
5. R. Courant and H. Robbins, *What is Mathematics*? Oxford, University Press, London-New York-Toronto, 1948.
6. A. N. Kostovskii, On the Possibility of Solving Construction Problems by Means of Compasses Alone with the Opening of the Legs Bounded, *Nauk. zap.*, Lvov University Press, 24. 1954 (in Russian).
7. A. N. Kostovskii, The Solution of Geometrical Construction Problems by Compasses Alone with a Fixed Angle, *Pit. Mekh. i Mat.*, Lvov University Press, 44, 1957 (in Ukrainian).
8. B. I. Kutuzov, *Geometry*, Uchpedgiz, Moscow, 1955 (in Russian).
9. L. Mascheroni, *La Geometria del Compasso*, Pavia, 1797.
10. G. Rademacher and Toeplitz, *Von Zahlen und Figuren*, Springer Verlag, Berlin, 1933.
11. I. F. Teslenko, *The Inversion Method and Its Applications*, Radyan'ska shkola, Kiev, 1954 (in Ukrainian).
12. A. M. Voronetz, *Geometry of the Circle*, Gostekhizdat, Moscow, 1934 (in Russian).
13. S. I. Zetel', *Geometry of the Line and Geometry of the Circle*, Uchpedgiz, Moscow, 1950 (in Russian).
14. K. Yanagihara, *On Limited Mascheroni Geometrical Construction*, Tôhoku Math. J., 1931, 34.

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This booklet is intended for a wide circle of readers. It should help teachers and pupils of senior classes of secondary schools to acquaint themselves in greater detail with geometrical constructions carried out by compasses alone. It can serve as a teaching aid in school mathematical clubs. The booklet can also be used by students of physical and mathematical departments of universities and teachers' training colleges to deepen their knowledge of elementary mathematics.

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